CSE 1400 Applied Discrete Mathematics
Polynomials

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Abstract

Polynomial functions are computationally primitive and form bases for approximating more complicated functions.

If f is a continuous real-valued function on $[a, b]$ and if any $\epsilon > 0$ is given, then there exists a polynomial p on $[a, b]$ such that $|f(x) - p(x)| < \epsilon$ for all x in $[a, b]$. In words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

Polynomials

Polynomials are important because

1. They are easy to evaluate.
2. They can approximate arbitrarily well many more complex functions.

Low degree polynomials $p(x)$ are studied in college algebra. College algebra studies how to solve polynomial equations $p(x) = 0$. and students learn to graph polynomial equations $p(x) = y$ in a Cartesian coordinate system.

Instances of polynomials include

$$p(x) = 3x + 2$$
$$p(x) = x^2 - x - 1$$
$$p(x) = x^2 - 4$$
$$p(x) = x^3 + x^2 + x + 1$$

The zeros of polynomials can be computed.

$$3x + 2 = 0 \quad \text{at } x = -2/3$$
$$x^2 - x - 1 = 0 \quad \text{at } x = \frac{1 \pm \sqrt{5}}{2}$$
$$x^2 - 4 = 0 \quad \text{at } x = \pm 2$$
$$x^3 + x^2 + x + 1 = 0 \quad \text{at } x = -1, \pm i$$

Constants

CONSTANTS ARE POLYNOMIALS. Constants are polynomials of degree 0. There are famous, important constants.

- 0, Zero
- 1, One
- $\sqrt{2} \approx 1.414213$, The square root of 2
- $\varphi = (1 + \sqrt{5})/2 \approx 1.618033$, The golden ratio
- $\pi \approx 3.141592$, pi
- $e \approx 2.718281$, Euler's or Napier's constant
- $\gamma \approx 0.577215$, The Euler-Mascheroni constant

Linear Polynomials

POLYNOMIALS THAT GROW BY A CONSTANT INCREMENT ARE CALLED LINEAR. A linear polynomial $p(x)$ is written

$$p(x) = mx + b$$

for some SLOPE $m \in \mathbb{R}$, $m \neq 0$ and y-INTERCEPT $b \in \mathbb{R}$. These functions are called first DEGREE polynomials.

The zero of the equation $p(x) = 0$ is $x = m^{-1}b$.

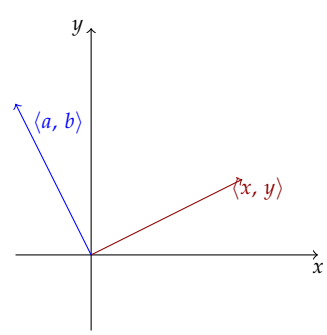
Lines through the origin are interesting. They have the form

$$ax + by = 0 \quad \text{where } (a \neq 0) \vee (b \neq 0)$$

One-dimensional affine space establishes an equivalence between two non-origin points (x_0, y_0) and (x_1, y_1) provided they lie on the same line through the origin. That is,

$$ax_0 + by_0 = 0 \quad \text{and} \quad ax_1 + by_1 = 0$$

$ax + by$ is the inner product of two vectors $\langle a, b\rangle$ and $\langle x, y\rangle$. When the inner product is 0, the vectors are ORTHOGONAL (perpendicular).



Quadratic Polynomials

QUADRATIC POLYNOMIALS ARISE IN MANY APPLICATIONS. A quadratic polynomial $p(x)$ is written

$$p(x) = ax^2 + bx + c \quad \text{where } a \neq 0$$

The SLOPE changes instantaneously as the function

$$p'(x) = 2ax + b$$

The zeros of the equation $p(x) = 0$ are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$.

And the y-INTERCEPT is $c \in \mathbb{R}$. These functions are called second DEGREE polynomials.

The golden quadratic is

$$\varphi(x) = x^2 - x - 1$$

It is named golden because one of its zeros, the values of x where $\varphi(x) = 0$, is the φ

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033$$

and the other is the conjugate root.

$$\overline{\varphi} = \frac{1-\sqrt{5}}{2} \approx -0.618033$$

The Power Basis

PRETEND A PROCESS CONSUMES $1 + y$ UNIT OF RESOURCES AND PRODUCES $1 + xy$ UNITS OF PRODUCT EACH PERIOD OF TIME. The process COMPOUNDS when the output is feed back as input during the next production cycle.

Assume that y is initially 0.

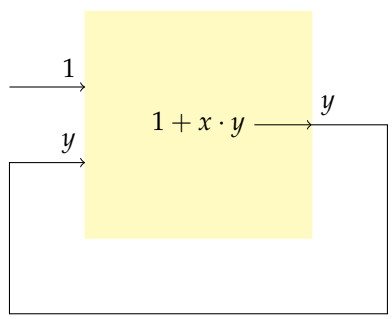


Table of y versus discrete time $n = 0, 1, 2, 3, \ldots$ shows how the feedback system operates. The polynomial

Table 1: Computing of y at rate x.

Time	0	1	2	3	4
y	0	1	$1+x$	$1+x+x^2$	$1+x+x^2+x^3$

$$p(x) = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$$

is called a geometric sum. You can prove a geometric sum is equal to the RATIONAL FUNCTION

$$p(x) = \frac{x^n - 1}{x - 1} \quad \text{for } x \neq 1$$

using mathematical induction.

For instance, consider $x = 2$. Each period you invest one unit and y, all of your previous investments. Your investment is worth $1 + 2y$ at the end of the period; the beginning of the next period. Let y_n be the value of y at time step n. Compute.

$$y_0 = \quad 0 \quad y_1 = \quad 1 \quad y_2 = \quad 3 \quad y_3 = 7$$

Did you discover the formula $y_n = 2^n - 1$? These are Mersenne numbers. They count the number of binary strings of length n.

The Binomial Theorem

THE BINOMIAL THEOREM STATES

$$\begin{aligned}(x+y)^n &= \sum_{0 \le k \le n} \binom{n}{k} x^{n-k} y^k \\ &= x^n + nx^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n-2} x^2 y^{n-2} + nxy^{n-1} + y^n\end{aligned}$$

Set $x = 1$ and $y = 1$ to derive the useful fact that 2^n is the sum of binomial coefficients in row n of Pascal's triangle. Interpret the binomial coefficient $\binom{n}{k}$ as the count of k-elements subsets over an n-element set to conclude there are 2^n subsets of an n-element set.

Horner's Rule for Evaluating Polynomials

HORNER'S RULE IS AN EFFICIENT ALGORITHM FOR EVALUATING A POLYNOMIAL $p(x)$ AT A GIVEN VALUE $x = c$. For instance, to evaluate $x^2 - x - 1$ at $x = 3$, write

HORNER'S RULE		
1	-1	-1
	3	6
1	2	$\boxed{5}$

For instance, to evaluate $3x^2 - 4x + 7$ at $x = -2$, write

HORNER'S RULE		
3	-4	7
	-6	20
3	-10	$\boxed{27}$

For instance, to evaluate $7x^5 - 2x^3 + 5x - 4$ at $x = 4$, write

Horner's Rule					
7	0	−2	0	5	−4
	28	112	440	1760	7020
7	28	110	440	1755	$\boxed{7016}$

Taylor Polynomials

Consider what happens when Horner's rule is iterated. For instance, for $\varphi(x) = x^2 - x - 1$ at $x = 3$, we find

Iterated Horner's Rule		
1	−1	−1
	3	6
1	2	$\boxed{5}$
	3	
1	$\boxed{5}$	
0		
$\boxed{1}$		

What this says is

$$\varphi(x) = x^2 - x - 1 = (x-3)^2 + 5(x-3)^1 + 5$$

which you can easily check.

For instance, we can write $\varphi(x) = x^2 - x - 1 = (x-a)^2 + b(x-a) + c$ for some shift a and coefficients b and c

$\varphi(a) = c$

Horner's Rule		
1	−1	−1
	a	$a(a-1)$
1	$a-1$	$\boxed{a^2-a-1}$
	a	
	$\boxed{2a-1}$	
$\boxed{1}$		

$$\varphi(x) = x^2 - x - 1 = (x-a)^2 + (2a-1)(x-a) + (a^2-a-1)$$

Falling Factorial Powers

The falling factorial powers

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1)$$

occur in some applications. The falling factorial powers are related to binomial coefficients by the identity.

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1) = \frac{n!x!}{n!(x-n)!} = n!\binom{x}{n}$$

The falling factorial powers are a natural basis for the sum and difference calculus. The fundamental theorem of the sum and difference calculus is the equation

$$\sum_{0\le k<n} x^{\underline{n}} = \frac{x^{\underline{n+1}}}{n+1}$$

This follows from the fact that the sum of values in column n of Pascal's triangle up to row x is equal to the value in the column $n+1$ and row $x+1$. That is

$$\sum_{0\le x<n} \binom{x}{n} = \binom{x+1}{n+1}$$
$$\sum_{0\le x<n} \frac{x^{\underline{n}}}{n!} = \frac{(x+1)^{\underline{n+1}}}{(n+1)!}$$

Stirling numbers of the second kind relate the polynomial power basis to falling factorial powers. That is,

$$\begin{aligned}
x^0 &= & x^{\underline{0}} &= 1\\
x^1 &= & x^{\underline{1}} &= x\\
x^2 &= & x^{\underline{2}} + x^{\underline{1}} &= x(x-1)+x\\
x^3 &= & x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} &= x(x-1)(x-2)+3x(x-1)+x\\
x^4 &= & x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} &= x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7(x-1)+x
\end{aligned}$$

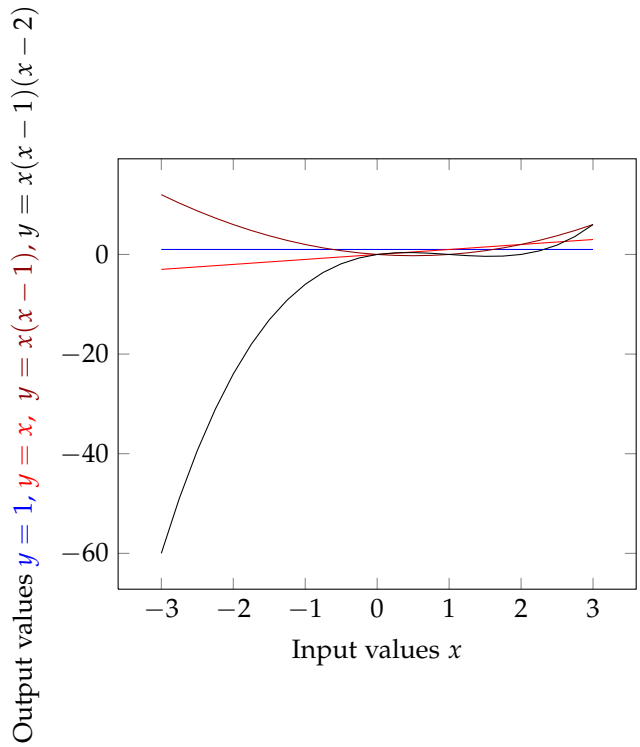
The general formula is

$$x^n = \sum_{0\le k\le n} \left\{ {n \atop k} \right\} x^{\underline{k}}$$

Where $\left\{ {n \atop k} \right\}$ are Stirling numbers of the second kind, defined by the recurrence

$$\left\{ {n \atop m} \right\} = \left\{ {n-1 \atop m-1} \right\} + m\left\{ {n-1 \atop m} \right\}$$

These are Stirling's subset numbers and they count the number of ways to partition an n element set into m subsets, that is they count equivalence relations.



Rising Factorial Powers

The rising factorial powers

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1)$$

occur in some applications. The rising factorial powers are related to binomial coefficients by the identity.

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1) = \frac{n!(x+n-1)!}{n!(x-1)!} = n!\binom{x+n-1}{n}$$

Stirling numbers of the first kind relate the polynomial power basis to rising factorial powers. That is,

$$\begin{aligned}
x^{\overline{0}} &= & x^0 &= 1\\
x^{\overline{1}} &= & x^1 &= x\\
x^{\overline{2}} &= & x^2 + x^1 &= x^2 + x\\
x^{\overline{3}} &= & x^3 + 3x^2 + 2x^1 &= x(x+1)(x+2)\\
x^{\overline{4}} &= & x^4 + 6x^3 + 11x^2 + 6x^1 &= x(x+1)(x+2)(x+3)
\end{aligned}$$

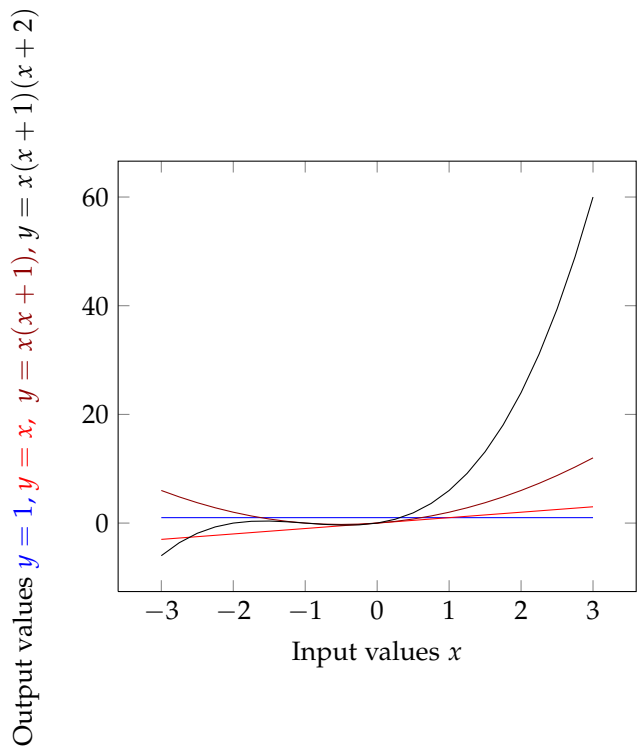
The general formula is

$$x^{\overline{n}} = \sum_{0\le k\le n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

Where $\left[{n \atop k}\right]$ are Stirling numbers of the first kind, defined by the recurrence

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix}$$

These are Stirling's cycle numbers and they count the number of ways permute n elements into m cycles.



Problems on Polynomials

1. Compute the degree of the polynomials
   (a) $p(x) = x^5 + 4x^3 + 2x - 7$
   (b) $p(x) = 5x^{10} - 1$
   (c) $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$
   (d) $p(x) = \sum_{0 \le k \le n-1} a_k x^k$
2. What are the roots of the polynomial equation $x^2 - x - 1 = 0$?
3. For $a \neq 0$, what are the roots of the polynomial equation $ax^2 + bx + c = 0$?
4. Answer the following `True` or `False`. Explain your answer.
   (a) $(1)(x-1) = x - 1$.
   (b) $(x+1)(x-1) = x^2 - 1$.
   (c) $(x^3 + x^2 + 1)(x-1) = x^3 - 1$.
   (d) $(x^{n-1} + x^{n-1} + \cdots + x + 1)(x-1) = x^n - 1$.

(e) $x^{n-1} + x^{n-2} + \cdots + x + 1 = (x^n - 1)/(x - 1)$ provided $x \neq 1$.

5. Use Horner's rule to evaluate the following polynomials at the given value of x.

(a) $p(x) = 3x^4 - 5x^2 - 16x - 4$ at $x = 3$.
(b) $p(x) = 3x^5 - 4x^3 + 3x^2 - 7$ at $x = 2$.
(c) $p(x) = -2x^4 - 3x^3 + 2x^2 + 4x - 3$ at $x = -3$.
(d) $p(x) = x^5 - 2x^3 + 3x^2 - 1$ at $x = -2$.
(e) $p(x) = x^4 + x^3 + x^2 + x + 1$ at $x = 2$.
(f) $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$ at $x = 2$.

6. Polynomials can be written in the falling (factorial) power basis $x^{\underline{0}} = 1$, $x^{\underline{1}} = x$, $x^{\underline{2}} = x(x-1)$, $x^{\underline{3}} = x(x-1)(x-2)$, $\ldots$, $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ as an alternative to the standard basis 1, x, x^2, x^3, $\ldots$, x^n.

(a) Horner's rule evaluates a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

written in the standard basis using n additions and n multiplies. Devise an algorithm to evaluate a polynomial

$$a_n x^{\underline{n}} + a_{n-1} x^{\underline{n-1}} + \cdots a_1 x^{\underline{1}} + a_0 x^{\underline{0}}$$

written in the falling power basis. Count the number of additions and multiplications your algorithm uses.

7. The set of all points (x, y) that lie on a line through the origin can be written in the form

$$ax + by = 0 \quad \text{where } (a \neq 0) \vee (b \neq 0)$$

(a) What is a logically equivalent way to write the clause $(a \neq 0) \vee (b \neq 0)$?
(b) Consider $\langle a, b \rangle$ to be the tip of a vector from the origin. What is the Euclidean length of $\langle a, b \rangle$ and how can you state the length is not zero?