1. For all (∀) and there exists (∃) are important quantifiers. ∀ allows you to say a predicate statement is always True or always False.

(∀ x)(p(x)) or (∀ x)(¬p(x))

∃ allows you to say a predicate statement is True or False for one or more values

(∃ x)(p(x)) or (∃ x)(¬p(x))

These quantifiers play together in strange and mysterious ways. It is worthwhile to learn the basics.

Consider the predicates below. Put (∀ x) in front of each of them and decide if the quantified statements is True or False. Do the same for (∃ x). I’ll use the notation :: to indicate how a predicate is defined. Assume the domain is the real numbers \( \mathbb{R} \).

(a) \( p(x) :: x = x + 1 \)
   Answer: Neither (∀ x)(x = x + 1) nor (∃ x)(x = x + 1) is True.

(b) \( p(x) :: x^2 - x - 1 = 0 \)
   Answer: (∀ x)(x^2 - x - 1 = 0) is False, but (∃ x)(x^2 - x - 1 = 0) is True. The roots of this polynomial equation are \( \phi = (1 + \sqrt{5})/2 \) and \( \bar{\phi} = (1 - \sqrt{5})/2 \), the golden ratio and its conjugate.

(c) \( p(x) :: x < 0 \)
   Answer: (∀ x)(x < 0) is False, but (∃ x)(x < 0) is True.

(d) \( p(x) :: 2^x > 0 \)
   Answer: (∀ x)(2^x > 0) is True and (∃ x)(2^x > 0) is True.

(e) Just a check that you are thinking: Can it be True that (∀ x)(p(x)) is True and (∃ x)(¬p(x)) is True?
   Answer: Well, no! If p(x) is True for all x’s, then there is no x for which it is False.

(f) Just a check that you are thinking: Do any of your answers change is you replace the domain \( \mathbb{R} \) with natural numbers, integers, or rational numbers: \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \).
   Answer: Well, yes! The answers for part (a) remains the same, but (b) is always False, (c) is False, False for domain \( \mathbb{N} \), but (c) is False, True for domain \( \mathbb{Z} \) and \( \mathbb{Q} \), and the answers for part (d) and (e) remain the same.

2. You know (∀ x)(∀ y) ≡ (∀ y)(∀ x) and (∃ x)(∃ y) ≡ (∃ y)(∃ x). But, in general (∀x)(∃ y) ≠ (∃ y)(∀ x)

Put (∀ x)(∀ y), (∃ x)(∃ y), (∀ x)(∃ y) and (∃ y)(∀ x) in front of each of these predicates and decide if the quantified expressions are True or False. Assume the domain is the real numbers \( \mathbb{R} \).

(a) \( p(x, y) :: x = y + 1 \)
   Answer:
   i. (∀ x)(∀ y)(x = y + 1) is False.
   ii. (∃ x)(∃ y)(x = y + 1) is True.
   iii. (∀ x)(∃ y)(x = y + 1) is True. Given x let y = x - 1.
   iv. (∃ y)(∀ x)(x = y + 1) is False.

(b) \( p(x, y) :: x^2 - x - 1 = y \)
   Answer:
i. \((\forall x)(\forall y)(x^2 - x - 1 = y)\) is False.
ii. \((\exists x)(\exists y)(x^2 - x - 1 = y)\) is True.
iii. \((\forall x)(\exists y)(x^2 - x - 1 = y)\) is True.
iv. \((\exists y)(\forall x)(x^2 - x - 1 = y)\) is False.

(c) \(p(x, y) :: x < y\)

Answer:

i. \((\forall x)(\forall y)(x < y)\) is False.
ii. \((\exists x)(\exists y)(x < y)\) is True.
iii. \((\forall x)(\exists y)(x < y)\) is True.
iv. \((\exists y)(\forall x)(x < y)\) is False.

(d) \(p(x, y) :: 2^x > y\)

Answer:

i. \((\forall x)(\forall y)(2^x > y)\) is False.
ii. \((\exists x)(\exists y)(2^x > y)\) is True.
iii. \((\forall x)(\exists y)(2^x > y)\) is True.
iv. \((\exists y)(\forall x)(2^x > y)\) is True.

3. Know your logarithms. Compute:

(a) \(\lg 256\)

Answer: \(\lg 256 = 8\) because \(2^8 = 256\).

(b) \(\lg 1/256\)

Answer: \(\lg 1/256 = -8\) because \(2^{-8} = 1/256\).

(c) \(\lg \sqrt[3]{2}\)

Answer: \(\lg \sqrt[3]{2} = 5/4\) because \(\lg \sqrt[3]{2} = 0.25 \lg 32\), or \(2^{5/4} = \sqrt[3]{2}\).

(d) \(\lg (0.25\sqrt{2})\)

Answer: \(\lg (0.25\sqrt{2}) = -3/2\) because \(\lg (0.25\sqrt{2}) = \lg 0.25 + \lg \sqrt{2} = -2 + \frac{1}{2}\).

(e) Write \(\log_b x\) in terms of \(\log_c x\).

Answer: The change of base rule for logarithms is

\[
\log_b x = \frac{\log_c x}{\log_c b}
\]

This is easy to derive: Let \(y = \log_b x\) so that \(b^y = x\). Compute the log base \(c\) of both sides and use the “log of a power is the power times the log” rule

\[
b^y = x
\]

\[
\log_c b^y = \log_c x
\]

\[
y \log_c b = \log_c x
\]

and solve for \(y\).

4. Use Horner’s rule to evaluate these polynomials at the given value of \(x\).

(a) \(p(x) = x^4 + x^3 + x^2 + x + 1\) at \(x = 2\). (How is this related to binary numbers?)

Answer:

\[
\begin{array}{cccccc}
\text{Horner’s Rule @ } x = 2 \\
1 & 1 & 1 & 1 & 1 \\
& 2 & 6 & 14 & 30 \\
\hline
1 & 3 & 7 & 15 & 31
\end{array}
\]

Therefore \(p(2) = 31\). This number can be interpreted as the binary number \((11111)_2\).
(b) $p(x) = 5x^4 + 7x^2 + 8x + 1$ at $x = -1$.

Answer:

<table>
<thead>
<tr>
<th>Horner’s Rule @ $x = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
</tr>
<tr>
<td>-5</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>

Therefore $p(-1) = 5$.

5. Use Horner’s rule to convert the following numbers to their decimal equivalent.

(a) $(10101010)_2$

Answer:

<table>
<thead>
<tr>
<th>Horner’s Rule @ $x = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Therefore, $(10101010)_2 = 170$

(b) $(1010.1010)_2$. Don’t use Horner’s rule again. Use what you just learned.

Answer: From the previous problem $(1010.1010)_2 = 170/16 = 10.625$

(c) $(BE)_{16}$

Answer:

<table>
<thead>
<tr>
<th>Horner’s Rule @ $x = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
</tr>
<tr>
<td>176</td>
</tr>
</tbody>
</table>

Therefore, $(BE)_{16} = 190$

(d) $(B.E)_{16}$. Don’t use Horner’s rule again. Use what you just learned.

Answer: From the previous problem $(B.E)_{16} = 190/16 = 11.7 = 11.875$.

6. Use repeated remaindering to convert the following numbers to their binary equivalent.

(a) 237

Answer:

<table>
<thead>
<tr>
<th>Quotients</th>
<th>237</th>
<th>118</th>
<th>59</th>
<th>29</th>
<th>14</th>
<th>7</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remainders</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore, $237 = (111010101)_2$.

(b) 2.37 (Expand to 5 bits after the binary point)

Answer: From the previous problem, you could divide $(11101101)_{10}$ by 100 = $(1100100)_2$, but that is rather unnatural. So use two steps

i. Convert 2 to $(10)_2$. 

3
ii. Convert 0.37 to binary by repeated multiplication by 2.

\[
\begin{align*}
0.37 \times 2 &= 0.74 \\
0.74 \times 2 &= 1.48 \\
0.48 \times 2 &= 0.96 \\
0.96 \times 2 &= 1.92 \\
0.92 \times 2 &= 1.84
\end{align*}
\]

Therefore, \(2.37 = (10.010111\cdots)_2\). Eventually, the pattern will repeat. I wonder how far you need to go before the repetition starts.