The Antisymmetric Property of Relations

The antisymmetric property is defined by a conditional statement.

Let $\sim$ be a relational symbol.

If ($x \sim y$ and $y \sim x$) implies $x = y$ for every $x, y \in \mathbb{U}$, then $\sim$ is antisymmetric.

Definition 1 (Antisymmetric Relation)**.** Let $\sim$ be a relation on set $\mathbb{U}$. The relation $\sim$ is antisymmetric if $x \sim y$ and $y \sim x$ implies $x = y$ for all $x, y \in \mathbb{U}$.

In symbols, $\sim$ is antisymmetric when

$$(\forall x, y \in \mathbb{U})(((x \sim y) \wedge (y \sim x)) \to x = y)$$

Representations of Antisymmetric Relations

If you recall, a relation is a set $\sim$ of ordered pairs. Antisymmetric says if $(x, y) \in \sim$ and $(y, x) \in \sim$ then $x = y$. Or by contraposition and De Morgan's law, if $x \neq y$, then $(x, y) \notin \sim$ or $(y, x) \notin \sim$.

Recall, a relation is an adjacency matrix. Antisymmetric says if the value in an off-diagonal entry say row j, column k is 1, then value in row k, column j is 0.

	a	b	c	d	$\cdots$	u
a	x	0	0	1	$\cdots$	1
b	0	x	1	0	$\cdots$	1
c	1	0	x	0	$\cdots$	0
d	0	0	1	x	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
u	0	0	1	0	$\cdots$	x

Examples of Antisymmetric Relations

The following relations are antisymmetric.

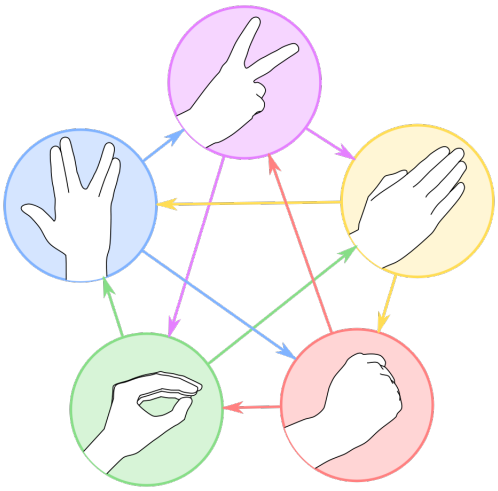
- Equality on the set of integers. $(\forall a, b \in \mathbb{Z})((a = b \wedge b = a) \to a = b)$.
- Less than or equal on the set of integers. $(\forall a, b \in \mathbb{Z})((a \leq b \wedge b \leq a) \to a = b)$.
- Divides on the set of natural numbers. $(\forall a, b \in \mathbb{N})((a \mid b \wedge b \mid a) \to (a = b)$.
- Proper subset on the power set of a universe $\mathbb{U}$. $(\forall \mathbb{X}, \mathbb{Y} \in 2^{\mathbb{U}})((\mathbb{X} \subset \mathbb{Y} \wedge \mathbb{Y} \subset \mathbb{X}) \to (\mathbb{X} = \mathbb{Y}))$.

The Rock–Paper–Scissors–Lizard–Spock Beats Relation

Consider the rules for Rock–Paper–Scissors–Lizard–Spock, where x beats y is given is given colorful names.

1. Scissors cuts paper
2. Paper covers rock
3. Rock crushes lizard
4. Lizard poisons Spock
5. Spock smashes scissors
6. Scissors decapitates lizard
7. Lizard eats paper
8. Paper disproves Spock
9. Spock vaporizes rock
10. Rock crushes scissors

The Rock–Paper–Scissors–Lizard–Spock Beats Relation



The Rock–Paper–Scissors–Lizard–Spock Beats Relation

The game's adjacency matrix shows the rules. Let R, P, Sc, L, Sp stand for Rock, Paper, Scissors, Lizard, Spock.

Rock-Paper-Scissors-Lizard-Spock

	R	P	Sc	L	Sp
R	1	0	1	1	0
P	1	1	0	0	1
Sc	0	1	1	1	0
L	0	1	0	1	1
Sp	1	0	1	0	1

In particular, you can verify the game is antisymmetric.

Relations That Are Not Antisymmetric

The following relations are <u>not</u> antisymmetric.

- Congruence mod n on the set of integers. A counterexample is,
$$(5 \equiv 14) \wedge (14 \equiv 5) \mod 3 \not\rightarrow 5 = 14$$
- Not equal on the integers. A counterexample is,
$$(1 \neq 2 \wedge 2 \neq 1) \not\rightarrow 1 = 2$$
- Odd products on the integers: A counterexample is,
$$(3 \cdot 5 \bmod 2 = 1) \wedge (5 \cdot 3 \bmod 2 = 1) \not\rightarrow 3 = 5$$

Counting Antisymmetric Relations

It is interesting to count antisymmetric relations on a finite set.

Theorem 2 (The Number of Antisymmetric Relations)**.** *Let $\mathbb{U}$ be a set with cardinality n. There are*
$$2^n 3^{n(n-1)/2} = \left(2\sqrt{3^{n-1}}\right)^n$$
symmetric relations on $\mathbb{U}$.

Recall, a relation on $\mathbb{U}$ can be represented as an $n \times n$ adjacency matrix A.

There are $n \times n = n^2$ entries in A, and each entries can take on one of 2 values.

So there are 2^{n^2} different adjacency matrices (relations).

Counting Antisymmetric Relations

Continuing from the previous slide.

To be antisymmetric, the matrix A can not have 1's in off-diagonal symmetric entries: if 1 is in row j, column k then 0 is in row k, column j.

Recall, there are $n(n-1)/2$ entries in the lower triangle of A. For each lower triangle entry $a(j, k)$, $j > k$ There are 3 possible assignments:

1. $a(j, k) = 0$ and $a(k, j) = 0$
2. $a(j, k) = 0$ and $a(k, j) = 1$
3. $a(j, k) = 1$ and $a(k, j) = 0$

There are $3^{n(n-1)/2}$ ways to fill off-diagonal entries in an antisymmetric matrix.

Counting Antisymmetric Relations

Continuing from the previous slide.

To completely fill in the adjacency matrix, the diagonal entries must be set. . There are 2 possible assignments for each of the n diagonal entries. The diagonal can be filled in 2^n ways. The number antisymmetric relation (adjacency matrices) is

$$2^n 3^{n(n-1)/2} = \left(2\sqrt{3^{n-1}}\right)^n$$

Related Relation

- A relation $\sim$ is <u>symmetric</u>, $x \sim y$, implies $y \sim x$. In symbols,

$$(\forall x, y \in \mathbb{U})((x \sim y) \to (y \sim x))$$

- A relation $\sim$ is <u>asymmetric</u> if for all x, y $((x \not\sim y) \vee (y \not\sim x))$. In symbols,

$$(\forall x, y \in \mathbb{U})((x \sim y) \to (y \not\sim x))$$

- A relation $\sim$ is <u>not antisymmetric</u> if there are values x and y such that $x \sim y$ and $y \sim x$ yet $x \neq y$. In symbols,

$$(\exists x, y \in \mathbb{U})((x \sim y) \wedge (y \sim x) \wedge (x \neq y))$$

Problems on Antiymmetric Relations

Abstract

The purpose of these exercises is to test your ability to recognize relations that are antisymmetric.

Which of the following are antisymmetric and which are not? Be able to explain your answer.

1. Equality on the integers.
2. Not equal on the integers.
3. Greater than on the integers.
4. Congruence mod n on the integers.
5. Divides on the natural numbers.