Cartesian Products

Definition 1 (Cartesian Product)**.** Let $\mathbb{X}$ and $\mathbb{Y}$ be sets.

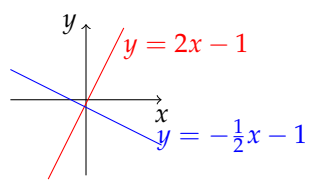
The Cartesian product of $\mathbb{X}$ by $\mathbb{Y}$ is

$$\mathbb{X} \times \mathbb{Y} = \{(x, y) : x \in \mathbb{X} \quad \text{and} \quad y \in \mathbb{Y}\}$$

Cartesian Products

The Cartesian product of $\mathbb{X}$ by $\mathbb{Y}$ is the set of all ordered pairs (x, y), where $x \in \mathbb{X}$ and $y \in \mathbb{Y}$.

You are familiar with drawing x and y axes and plotting functions in this plane.



The Cartesian plane is a geometric representation of the Cartesian product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Examples of the Cartesian Product

Let $\mathbb{B} = \{0, 1\}$ be the bits. Then the Cartesian product $\mathbb{B} \times \mathbb{B}$ is

$$\mathbb{B} \times \mathbb{B} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

Let $\mathbb{A} = \{a, b, c\}$. Then the Cartesian product $\mathbb{B} \times \mathbb{A}$ is

$$\mathbb{B} \times \mathbb{A} = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$$

And, $\mathbb{A} \times \mathbb{B}$ is

$$\mathbb{A} \times \mathbb{B} = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$$

The Cartesian product is not commutative.

Cardinality of Cartesian Products

If $\mathbb{X}$ is a set with cardinality n ($|\mathbb{X}| = n$).

And $\mathbb{Y}$ is a set with cardinality m ($|\mathbb{Y}| = m$).

Then the Cartesian product

$$\mathbb{X} \times \mathbb{Y} = \{(x, y) : x \in \mathbb{X} \quad \text{and} \quad y \in \mathbb{Y}\}$$

has cardinality

$$|\mathbb{X} \times \mathbb{Y}| = n \times m = nm.$$

Binary Relations

A subset of the Cartesian product $\mathbb{X} \times \mathbb{Y}$ is called a <u>relation</u> from $\mathbb{X}$ to $\mathbb{Y}$.

Rather than write relations as sets, they are most often written using a relational symbol.

For instance

- $x = y$ (x equals y)
- $x < y$ (x is less than y)
- $x \mid y$ (x divides y)

Counting Binary Relations

Let $\mathbb{X}$ and $\mathbb{Y}$ be finite sets with $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$.

Then, the cardinality of $\mathbb{X} \times \mathbb{Y}$ is nm.

And there are 2^{nm} subsets of $\mathbb{X} \times \mathbb{Y}$.

Since every subset is a relation, there are 2^{nm} relations from $\mathbb{X}$ to $\mathbb{Y}$.

Theorem 2 (Number of Relations)**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be sets with cardinalities n and m, respectively. There are 2^{nm} relations from $\mathbb{X}$ to $\mathbb{Y}$.*