Cartesian Products

Definition 1 (Cartesian Product). Let $X$ and $Y$ be sets. The Cartesian product of $X$ by $Y$ is

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

Cartesian Products

The Cartesian product of $X$ by $Y$ is the set of all ordered pairs $(x, y)$, where $x \in X$ and $y \in Y$.

You are familiar with drawing $x$ and $y$ axes and plotting functions in this plane.

The Cartesian plane is a geometric representation of the Cartesian product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Examples of the Cartesian Product

Let $B = \{0, 1\}$ be the bits. Then the Cartesian product $B \times B$ is

$$B \times B = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

Let $A = \{a, b, c\}$. Then the Cartesian product $B \times A$ is

$$B \times A = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$$

And, $A \times B$ is

$$A \times B = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$$

The Cartesian product is not commutative.

Cardinality of Cartesian Products

Let $X$ is a set with cardinality $n$ ($|X| = n$).

And $Y$ is a set with cardinality $m$ ($|Y| = m$).

Then the Cartesian product

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

has cardinality

$$|X \times Y| = n \times m = nm.$$
Binary Relations

A subset of the Cartesian product $X \times Y$ is called a relation from $X$ to $Y$.

Rather than write relations as sets, they are most often written using a relational symbol.

For instance
- $x = y$ (x equals y)
- $x < y$ (x is less than y)
- $x \mid y$ (x divides y)

Counting Binary Relations

Let $X$ and $Y$ be finite sets with $|X| = n$ and $|Y| = m$.

Then, the cardinality of $X \times Y$ is $nm$.

And there are $2^{nm}$ subsets of $X \times Y$.

Since every subset is a relation, there are $2^{nm}$ relations from $X$ to $Y$.

**Theorem 2 (Number of Relations).** Let $X$ and $Y$ be sets with cardinalities $n$ and $m$, respectively. There are $2^{nm}$ relations from $X$ to $Y$. 