Counting Functions by Subsets

It is useful to be able to count the number of functions from a finite set to another finite set.

One way to count functions is by recognizing that a function can be represented as a set of ordered pairs with a special property

\((\forall x \in X)(\exists! y \in Y)((x, y) \in f)\)

Let \(f : X \to Y\) be a function. Let \(|X| = n\) and \(|Y| = m\).

Then \(f\) is a subset of the Cartesian product \(X \times Y\). And the cardinality of \(f\) is \(n\).

For each of the \(n\) pairs \((x, y) \in f\) there are \(m\) possible ways to fill in the \(y\) value. Therefore, there are

\(m^n\) functions \(f : X \to Y\)

Counting Functions by Adjacency Matrix

Another way to count functions is by recognizing that a function can be represented by an adjacency matrices. Let \(f : X \to Y\) be a function, and let \(|X| = n\) and \(|Y| = m\). Then \(f\) is an \(n \times m\) adjacency matrix.

\[
\begin{array}{cccccc}
 & 0 & 1 & 2 & \cdots & m - 2 & m - 1 \\
\hline
n & 0 & 1 & & & & \\
\hline
\text{rows} & & & & & & n - 1 \\
\end{array}
\]

Each row will have a 1 in one and only one column. There are \(m\) column choices for each of the \(n\) rows.

Therefore, there are \(m^n\) functions from \(X\) to \(Y\).
Counting Functions by Adjacency Matrix
As a "counting functions" example, let

\[ X = \{0, 1\} \quad \text{and} \quad Y = \{a, b, c\} \]

There are \(3^2 = 9\) functions from \(X\) to \(Y\).

1. \[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}
\]

2. \[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

3. \[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

4. \[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}
\]

5. \[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

6. \[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

7. \[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0
\end{pmatrix}
\]

8. \[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

9. \[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

Counting Functions by Bipartite Graph
Another way to count functions is by recognizing that a function can be represented by a bipartite graph.

Let \(f : X \to Y\) be a function. Let \(|X| = n\) and \(|Y| = m\).

Then, \(f\) is a bipartite graph from \(X\) to \(Y\).

A bipartite graph is a collection of directed edges from \(X\) to \(Y\).

To be a function, the graph has one and only one edge leaving each element in \(X\).

That edge can be directed at any of the \(m\) elements in \(Y\).

Therefore, there are \(m^n\) functions from \(X\) to \(Y\).

Counting Functions by Adjacency Matrix
As a "counting functions" example, let

\[ X = \{0, 1\} \quad \text{and} \quad Y = \{a, b, c\} \]

There are \(3^2 = 9\) functions from \(X\) to \(Y\).
Theorem 1 (Counting Functions). Let the cardinality of \( X \) be \( n \) (\( |X| = n \)) and let the cardinality of \( Y \) be \( n \) (\( |Y| = m \)).

Then there are \( m^n \) functions from \( X \) to \( Y \)