Exponential Functions

Exponential functions are interesting and useful.

The general exponential function has the form

\[ y = \exp(x; a, b) = ab^x \]

where the parameters are: the base \( b > 0 \), \( b \neq 1 \) and the \( y \)-intercept \( a \).

Let \( a = 1 \), then the exponential function satisfies the rules:

- \( b^{x+t} = b^x b^t \)
- \( b^{x-t} = b^x / b^t \)
- \( (b^x)^t = b^{tx} \)

The Primary Bases for Exponential Functions

There are three primary bases for exponential functions.

1. Base 10: \( y = 10^x \) useful because of the common way numbers are written.
2. Base \( e \approx 2.71828 \): \( y = e^x \) useful in analysis for reasons beyond the scope of this class.
3. Base 2: \( y = 2^x \) useful in computing: representation of numbers and data, divide and conquer algorithm analysis, etc.

Applications of Exponential Functions

Moore's law: The performance of integrated circuits roughly doubles every 2 years.

That is, the performance of integrated circuits is roughly given by

\[ P(t) = 2^{t/2} \]

where time \( t \) is measured in years.

Notice

\[ P(t + 2) = 2^{(t+2)/2} = 2^{t/2+1} = 2P(t) \]
Applications of Exponential Functions

Malthus proposed that populations grow at an exponential rate. That is,

\[ P(t) = P_0 e^{rt} \]

where \( P_0 \) is the initial population, \( r \) is the growth rate and \( t \) is time in years.

You can find the time for a population to double by solving

\[ e^{rt} = 2 \]

for \( t \)

Which gives \( t = \ln(2)/r \approx 70/100 \) called the rule of 70.

Legends of Exponential Growth

The Tower of Hanoi legend states the world will end after the monks complete \( 2^{64} - 1 \) moves of 64 golden disks; a very long time, even for very fast monks.

The legends of Rani and the Raja tells the story of a peasant girl who befriended the ruler. As a reward, Rani asked for only 1 grain of rice, to be doubled each day for a month. With her reward, Rani feed all the hungry people. Rani had accumulated

\[ 1 + 2 + 4 + \cdots + 2^{30} = 2^{31} - 1 \] grains of rice.

One estimate is there are about \( 2^{12.6} \) grains of rice in a cup, which feeds a person for a day. Rani could feed a about \( 2^{18.4} \approx 345,900 \) people for a day.

Logarithm Functions

Logarithm functions are interesting and useful.

The general logarithm function has the form

\[ y = \log_b(x; c) = \log_b(x) + c \]

where the parameters are: the base \( b > 0, b \neq 1 \) and the \( x \)-intercept \( 1/b^c \).

Let \( c = 0 \). The logarithm function satisfies the rules:

- \( \log_b(xt) = \log_b x + \log_b t \)
- \( \log_b(x/t) = \log_b x - \log_b t \)
- \( \log_b x^t = t \log_b x \)
The Primary Bases for Logarithm Functions

There are three primary bases for logarithm functions.

1. Base 10: \( y = \log_{10} x = \log x \) useful for introducing learners to the concept of logarithms.

2. Base \( e \approx 2.71828 \): \( y = \log_e x = \ln x \) useful in analysis for reasons beyond the scope of this class.

3. Base 2: \( y = \log_2 x = \lg x \) useful in computing: representation of numbers and data, divide and conquer algorithm analysis, etc.

Applications of Logarithm Functions

1. Moore’s law: The performance of integrated circuits roughly doubles every 2 years. That is, the performance of ICs is \( P(t) = 2^{t/2} \), where time \( t \) is measured in years. Notice

   \[
   P(t + 2) = 2^{(t+2)/2} = 2^{t/2+1} = 2P(t)
   \]

2. Malthus proposed that populations grow at an exponential rate. That is,

   \[
   P(t) = P_0e^{rt}
   \]

   where \( P_0 \) is the initial population, \( r \) is the growth rate and \( t \) is time in years.

   You can find the time for a population to double by solving

   \[
   e^{rt} = 2 \quad \text{for } t
   \]

   Which gives \( t = \ln 2/r \approx 70/100r \) called the rule of 70.

Graphs of the Logarithm and Exponential Functions

![Graphs of the Logarithm and Exponential Functions](image)