One-to-One Functions

Definition 1 (One-to-One Function). A function \( f : X \rightarrow Y \) is one-to-one if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \), or equivalently, \( x_1 \neq x_2 \) implies \( f(x_1) \neq f(x_2) \).

Each \( y \) in \( Y \), is the image of at most one \( x \) in \( X \).

\[
(\forall x_1, x_2 \in Y)((f(x_1) = f(x_2)) \rightarrow (x_1 = x_2))
\]
or

\[
(\forall x_1, x_2 \in Y)((x_1 \neq x_2) \rightarrow (f(x_1) \neq f(x_2)))
\]

One-to-One Functions

Recall, that a function can be represented by a graph, a good way to visualize the one-to-one idea.

Here is an example showing an one-to-one function.

[Graph of a one-to-one function with points 0, 1, 2, 3, 4, 5 mapping to a, b, c, d, e, f, respectively.]

One-to-One Functions

And, here is an example function that is not one-to-one.

[Graph of a non-one-to-one function with points 0, 1, 2, 3, 4, 5 mapping to a, b, c, d, e, f, respectively.]

\( b \) is the images of three values in \( X \) and \( d \) is the images of both 4 and 5.
The Horizontal Line Test for One-to-One-ness

When a function \( y = f(x) \) is graphed using Cartesian coordinates in the traditional analytic geometry fashion, the function is one-to-one when every horizontal line crosses its graph at most once.

When some horizontal line crosses the graph of \( y = f(x) \), more than once the \( f \) is not one-to-one.

Example One-to-One Functions

The following are one-to-one functions.

- All polynomials of degree 1 are one-to-one. For instance, \( f(x) = 3x + 2 \) is one-to-one.
  
  To see this, if \( 3x_1 + 2 = 3x_2 + 2 \), then \( x_1 = x_2 \).

- The common (base 10) logarithm \( f(x) = \log x \) is one-to-one.
  
  To see this, if \( \log x_1 = \log x_2 \), then \( \log x_1 - \log x_2 = 0 \) and so \( \log(x_1/x_2) = 0 \) which implies \( x_1/x_2 = 1 \).

Example Functions That Are Not One-to-One

The following are not one-to-one functions.

- \( f(x) = 3x^2 + 2 \) is not one-to-one.
  
  To see this, notice \( f(-1) = 5 \) and \( f(1) = 5 \).

- \( f(n) = n \mod 3 \) is not one-to-one. \( 1 \mod 3 = 4 \mod 3 = 7 \mod 3 = \cdots \).

One-to-One Depends on Cardinality of Domain and Co-domain

Let \( |X| = n \) and \( |Y| = m \) be the cardinalities of \( X \) and \( Y \).

Recall, the cardinality is the number of elements in the set.

If \( Y \) has fewer elements than \( X \), that is, if \( |Y| < |X| \), then it is not possible to define an one-to-one (total) function from \( X \) to \( Y \).
No One-to-One Functions from Large to Smaller Sets

You cannot have an one-to-one function from a large set to a smaller set.

On the other hand, a small domain does not guarantee a one-to-one function.