**Partial Orders**

A common task is to arrange events in order.

Precedence (before or after) are scheduling ideas.

The precedent symbol $\prec$ can be used to denote that one event precedes another and $\preceq$ allows for events that occur at the same time.

You are familiar with certain orders:

- Less than $<$, or, Less than or equal $\leq$
- Proper subset $\subset$, or, subset $\subseteq$.

**Precedence Order for Operations**

You are familiar with precedence order for arithmetic operations.

Exponentiate before multiplying and multiply before adding. (Of course, parentheses can be used to override precedence)

$$a^b \prec a \cdot b \prec a + b$$

And there is a precedence for Boolean operations.

Negate before conjuncting and conjunct before disjuncting and disjunct before implying.

$$\neg P \prec P \wedge Q \prec P \vee Q \prec P \rightarrow Q$$

Computer programming languages implement many operations and have an established precedence for evaluating expressions.

**Partial Orders**

Ask yourself, what could you say about events that can be placed in order?

Let $a$, $b$, and $c$ be events in some universe $\mathbb{U}$.

- Event $a$ occurs at the same time as itself : $a \preceq a$. Partial orders are <u>reflexive</u>. (<u>Strict orders</u> $a \prec a$ are irreflexive: $a \prec a$ is `False` for all events $a$.)
- If $a$ precedes $b$ and $b$ precedes $a$, then $a$ and $b$ occur at the same time. If $a \preceq b$ and $b \preceq a$ , then $a = b$. partial orders are <u>antisymmetric</u>.
- If $a$ precedes $b$ and $b$ precedes $c$, then $a$ precedes $c$. if $a \preceq b$ and $b \preceq c$, then $a \preceq c$. partial orders are <u>transitive</u>.

**Partial Orders**

**Definition 1** (Partial Order)**.** A <u>partial order</u> obeys three properties:

1. Reflexive: $(\forall a)(a \preceq a)$
2. Antisymmetric: $(\forall a, b)(((a \preceq b) \wedge (b \preceq a)) \rightarrow (a = b))$
3. Transitive: $(\forall a, b, c)(((a \preceq b) \wedge (b \preceq c)) \rightarrow (a \preceq c))$

**Subset is a Partial Orders**

Subset is a partial order.

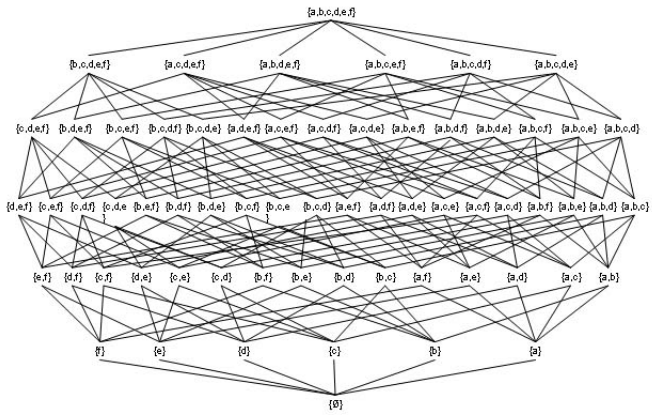
1. $\mathbb{A} \subseteq \mathbb{A}$ for every set $\mathbb{A}$.
2. If $\mathbb{A} \subseteq \mathbb{B}$ and $\mathbb{B} \subseteq \mathbb{A}$ then $\mathbb{A} = \mathbb{B}$.
3. If $\mathbb{A} \subseteq \mathbb{B}$ and $\mathbb{B} \subseteq \mathbb{C}$ then $\mathbb{A} \subseteq \mathbb{C}$.

Subsets can be placed in order.

$$\begin{tikzcd}
 & \emptyset \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\{0\} \arrow[dr] \arrow[drr] & & \{1\} \arrow[dl] \arrow[dr] & \{2\} \arrow[dl] \arrow[dr] \\
 & \{0, 1\} \arrow[drr] & \{0, 2\} \arrow[dr] & \{1, 2\} \arrow[d] \\
 & & & \{0, 1, 2\}
\end{tikzcd}$$

**Subset is a Partial Orders**

This image showing subset ordering on a 6 element set was created by Gina Cook and downloaded from Wikipedia.




**Total Orders**

You will have noticed that not all subsets can be compared to each other. $\{0\}$ is not a subset of $\{1\}$, and vice versa.

That is why the order is called partial.

When every pair of objects can be placed in order, the relation is called a total order.

For instance less than or equal on the integers is a total order.

**Counting Transitive Relations**

There is no simple formula for counting partial or total orders.

However, the number of partial and total orders can be computed, see The Encyclopedia of Integer Sequences: A001035 and Sequence A000670 for partial orders and total orders.

| Set Cardinality | Count of Partial Orders | Count of Total Orders |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 1 |
| 2 | 3 | 2 |
| 3 | 19 | 6 |
| 4 | 219 | 24 |
| 5 | 4231 | 120 |

**Problems on Orders**

Show your understanding of this topic by completing the problems found at Orders.