Partial Orders

A common task is to arrange events in order. Precedence (before or after) are scheduling ideas.
The precedent symbol $\prec$ can be used to denote that one event precedes another and $\preceq$ allows for events that occur at the same time.
You are familiar with certain orders:
- Less than $<$, or, Less than or equal $\leq$
- Proper subset $\subset$, or, subset $\subseteq$.

Precedence Order for Operations

You are familiar with precedence order for arithmetic operations.
Exponentiate before multiplying and multiply before adding. (Of course, parentheses can be used to override precedence)

$$a^b \prec a \cdot b \prec a + b$$

And there is a precedence for Boolean operations.
Negate before conjuncting and conjunct before disjuncting and disjunct before implying.

$$\neg p \prec p \land q \prec p \lor q \prec p \rightarrow q$$

Computer programming languages implement many operations and have an established precedence for evaluating expressions.

Partial Orders

Ask yourself, what could you say about events that can be placed in order?
Let $a$, $b$, and $c$ be events in some universe $U$.

- Event $a$ occurs at the same time as itself: $a \preceq a$. Partial orders are reflexive. (Strict orders $a \prec a$ are irreflexive: $a \prec a$ is False for all events $a$.)
- If $a$ precedes $b$ and $b$ precedes $a$, then $a$ and $b$ occur at the same time. If $a \preceq b$ and $b \preceq a$, then $a = b$. Partial orders are antisymmetric.
- If $a$ precedes $b$ and $b$ precedes $c$, then $a$ precedes $c$. If $a \preceq b$ and $b \preceq c$, then $a \preceq c$. Partial orders are transitive.
Partial Orders

**Definition 1 (Partial Order).** A partial order obeys three properties:

1. Reflexive: $\forall a (a \preceq a)$
2. Antisymmetric: $\forall a, b (((a \preceq b) \land (b \preceq a)) \rightarrow (a = b))$
3. Transitive: $\forall a, b, c (((a \preceq b) \land (b \preceq c)) \rightarrow (a \preceq c))$

Subset is a Partial Orders

**Subset is a partial order:**

1. $A \subseteq A$ for every set $A$.
2. If $A \subseteq B$ and $B \subseteq A$ then $A = B$.
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
Subsets can be placed in order.

![Diagram of partial order with subsets {0}, {0, 1}, {0, 2}, {1, 2}, {2} and their relationships.]}
Subset is a Partial Orders

This image showing subset ordering on a 6 element set was created by Gina Cook and downloaded from Wikipedia.

Total Orders

You will have noticed that not all subsets can be compared to each other. \{0\} is not a subset of \{1\}, and vice versa.

That is why the order is called partial.

When every pair of objects can be placed in order, the relation is called a total order.

For instance less than or equal on the integers is a total order.

Counting Transitive Relations

There is no simple formula for counting partial or total orders.

However, the number of partial and total orders can be computed, see The Encyclopedia of Integer Sequences: A001035 and Sequence A000670 for partial orders and total orders.

<table>
<thead>
<tr>
<th>Set Cardinality</th>
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<th>Count of Total Orders</th>
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<tr>
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Problems on Orders

Show your understanding of this topic by completing the problems found at Orders.