The Reflexive Property of Relations

When every element of a set is related to itself, the relation is <u>reflexive</u>.

Definition 1 (Reflexive Relation)**.** A relation $\sim$ on $\mathbb{U}$ is <u>reflexive</u> if $x \sim x$ for all $x \in \mathbb{U}$.

In symbols, $\sim$ is reflexive when

$$(\forall x \in \mathbb{U})(x \sim x)$$

Representations of Reflexive Relations

If you recall, a relation is a set $\sim$ of ordered pairs. <u>Reflexive</u> says $(u, u) \in \sim$ for all $u \in \mathbb{U}$.

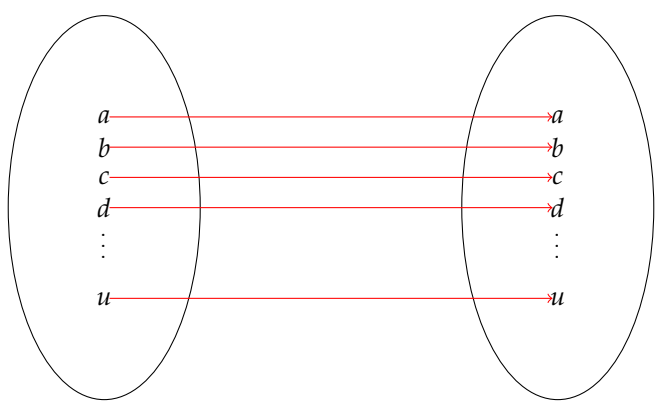
If you recall, a relation is an adjacency matrix. <u>Reflexive</u> says 1 is the value of each entry along the main diagonal of the matrix.

	a	b	c	d	$\cdots$	u
a	1	x	x	x	$\cdots$	x
b	x	1	x	x	$\cdots$	x
c	x	x	1	x	$\cdots$	x
d	x	x	x	1	$\cdots$	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
u	x	x	x	x	$\cdots$	1

The values in off-diagonal entries do not effect reflexivity.

Representations of Reflexive Relations

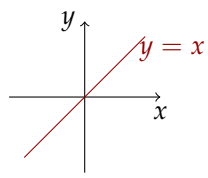
If you recall, a relation is a Bipartite Graph. <u>Reflexive</u> says there is a directed edge from u to u for all $u \in \mathbb{U}$.



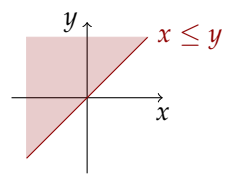
Examples of Reflexive Relations

The following relations are reflexive.

- Equality on the real numbers. $(\forall a \in \mathbb{R})(a = a)$.



- Less than or equal on the set of integers. $(\forall a \in \mathbb{Z})(a \leq a)$.



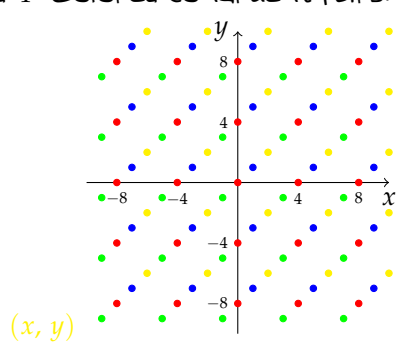
Examples of Reflexive Relations

Congruence mod n on the integers is <u>reflexive</u>.

$$(\forall a \in \mathbb{Z})(a \equiv a \pmod{n}) \quad (a - a = 0 = n \cdot 0)$$

In the diagram below, the diagonal $y = x$ is covered.

Congruence mod 4 Colored congruent pairs: (x, y), (x, y), (x, y),

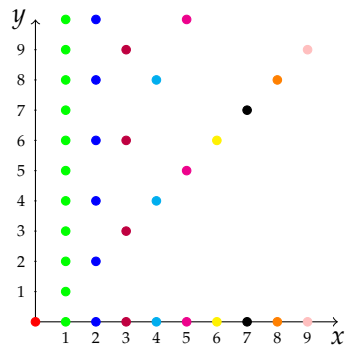


Examples of Reflexive Relations

Divides on the natural numbers is <u>reflexive</u>.

$$(\forall a \in \mathbb{Z})(a \mid a) \quad (a \cdot 1 = a)$$

In the diagram below, the diagonal $y = x$ is covered.



Divides on the natural numbers $\mathbb{N}$

Examples of Reflexive Relations

Conditional statements $\mathsf{p} \to \mathsf{q}$ on $\mathbb{B}$. $(\forall \mathsf{p} \in \mathbb{B})(\mathsf{p} \to \mathsf{p})$.

$$(\texttt{False} \to \texttt{False}) \equiv \texttt{True}$$

$$(\texttt{True} \to \texttt{True}) \equiv \texttt{True}$$

Examples of Relations that are Not Reflexive

The following relations are <u>not</u> reflexive.

- Less than (strict inequality) on the set of integers. A counterexample is $1 < 1$ is a `False` statement.
- Not equal on the set of integers. A counterexample is $1 \neq 1$ is a `False` statement.
- Proper subset on $2^{\mathbb{N}}$, the power set $\mathbb{N}$. A counterexample is $\{0\} \subset \{0\}$ is a `False` statement.

Counting Reflexive Relations

It is interesting to count reflexive relations on a finite set.

Theorem 2 (The Number of Reflexive Relations)**.** *Let $\mathbb{U}$ be a set with cardinality n. There are $2^{n(n-1)}$ reflexive relations on $\mathbb{U}$.*

A relation on $\mathbb{U}$ can be represented as an $n \times n$ adjacency matrix A.

There are $n \times n = n^2$ entries in A, and each entries can take on one of 2 values.

To be reflexive, the matrix A must have a 1 (`True`) value along its main diagonal.

Using Adjacency Matrices to Count Reflexive Relations

Continuing from the previous slide.

The off-diagonal entries of A can be either 0 or 1.

The number of entries in the the lower triangle of A is the triangular number

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

For instance, there are $10 = 5(5-1)/2$ entries in the lower triangle of a 5×5 matrix

	a	b	c	d	e
a	1				
b	7	1			
c	4	8	1		
d	2	5	9	1	
e	1	3	6	10	1

Using Adjacency Matrices to Count Reflexive Relations

Continuing from the previous slide.

There are twice $n(n-1)/2$ or $n(n-1) = n^2 - n$ off-diagonal entries.

Each of these $n(n-1)$ entries can can be a 0 or a 1.

So there are $2^{n(n-1)}$ reflexive adjacency matrices.

Counting Reflexive Relations

Let $\mathbb{U}$ be a set with cardinality n..

A relation on $\mathbb{U}$ can be represented as a subset $\sim$ of the Cartesian product $\mathbb{U} \times \mathbb{U}$.

There are $n \times n = n^2$ ordered pairs in $\mathbb{U} \times \mathbb{U}$.

To be reflexive, the subset $\sim$ must contain (u, u) for each $u \in \mathbb{U}$. That is, these n ordered pairs must be related.

There are $n^2 - n = n(n-1)$ other ordered pairs that can be in or out of $\sim$.

So there are $2^{n(n-1)}$ subsets that represent reflexive relations.

Related Relation

- A relation $\sim$ is <u>not reflexive</u> if there is a value x such that $x \not\sim x$. In symbols,
$$(\exists x \in \mathbb{U})(x \not\sim x)$$
The relation "xy is even" on the natural numbers is not reflexive. For instance, $1 \cdot 1$ is not even.

- A relation $\sim$ is <u>irreflexive</u> if $x \not\sim x$ for all x. In symbols,
$$(\forall x \in \mathbb{U})(x \not\sim x)$$
Not equal and less than are both irreflexive.

  For all values of x, $x \neq x$ and $x \not< x$ both `True`.

Problems on Reflexive Relations

Show your understanding of this topic by completing the problems found at Reflexive Relations.