The Reflexive Property of Relations

When every element of a set is related to itself, the relation is reflexive.

Definition 1 (Reflexive Relation). A relation \( \sim \) on \( U \) is reflexive if \( x \sim x \) for all \( x \in U \).

In symbols, \( \sim \) is reflexive when

\[
(\forall x \in U)(x \sim x)
\]

Representations of Reflexive Relations

If you recall, a relation is a set \( \sim \) of ordered pairs. Reflexive says

\[(u, u) \in \sim \text{ for all } u \in U.\]

If you recall, a relation is an adjacency matrix. Reflexive says 1 is the value of each entry along the main diagonal of the matrix

\[
\begin{array}{ccccccc}
  & a & b & c & d & \cdots & u \\
 a & 1 & x & x & x & \cdots & x \\
b & x & 1 & x & x & \cdots & x \\
c & x & x & 1 & x & \cdots & x \\
d & x & x & x & 1 & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u & x & x & x & x & \cdots & 1 \\
\end{array}
\]

The values in off-diagonal entries do not affect reflexivity.

Representations of Reflexive Relations

If you recall, a relation is a bipartite graph. Reflexive says there is a directed edge from \( u \) to \( u \) for all \( u \in U \).
Examples of Reflexive Relations

The following relations are reflexive.

- Equality on the real numbers. $(\forall a \in \mathbb{R})(a = a)$.

- Less than or equal on the set of integers. $(\forall a \in \mathbb{Z})(a \leq a)$.

Examples of Reflexive Relations

Congruence mod $n$ on the integers is reflexive.

$(\forall a \in \mathbb{Z})(a \equiv a \pmod{n}) \quad (a - a = 0 = n \cdot 0)$

In the diagram below, the diagonal $y = x$ is covered.

Congruence mod 4 Colored congruent pairs: $(x, y), (x, y), (x, y), (x, y)$,
Examples of Reflexive Relations

Divides on the natural numbers is reflexive.

\[(\forall a \in \mathbb{Z})(a \mid a) \quad (a \cdot 1 = a)\]

In the diagram below, the diagonal \(y = x\) is covered.

\[\text{Divides on the natural numbers } \mathbb{N}\]

Examples of Reflexive Relations

Conditional statements \(p \rightarrow q\) on \(\mathbb{B}\). \((\forall p \in \mathbb{B})(p \rightarrow p)\).

\[(\text{False} \rightarrow \text{False}) \equiv \text{True}
\]

\[(\text{True} \rightarrow \text{True}) \equiv \text{True}\]

Examples of Relations that are Not Reflexive

The following relations are not reflexive.

- Less than (strict inequality) on the set of integers. A counterexample is \(1 < 1\) is a False statement.

- Not equal on the set of integers. A counterexample is \(1 \neq 1\) is a False statement.

- Proper subset on \(2^\mathbb{N}\), the power set \(\mathbb{N}\). A counterexample is \(\{0\} \subset \{0\}\) is a False statement.
Counting Reflexive Relations
It is interesting to count reflexive relations on a finite set.

**Theorem 2** (The Number of Reflexive Relations). Let $\mathcal{U}$ be a set with cardinality $n$. There are $2^{n(n-1)}$ reflexive relations on $\mathcal{U}$.

A relation on $\mathcal{U}$ can be represented as an $n \times n$ adjacency matrix $A$.

There are $n \times n = n^2$ entries in $A$, and each entries can take on one of 2 values.

To be reflexive, the matrix $A$ must have a 1 (True) value along its main diagonal.

Using Adjacency Matrices to Count Reflexive Relations
Continuing from the previous slide.

The off-diagonal entries of $A$ can be either 0 or 1.

The number of entries in the the lower triangle of $A$ is the triangular number

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

For instance, there are $10 = 5(5-1)/2$ entries in the lower triangle of a $5 \times 5$ matrix

$$
\begin{array}{ccccc}
| & a & b & c & d & e \\
\hline
a & 1 & & & & \\
b & 1 & 1 & & & \\
c & & 8 & 1 & & \\
d & & & 9 & 1 & \\
e & & & & 1 & \\
\end{array}
$$

Using Adjacency Matrices to Count Reflexive Relations
Continuing from the previous slide.

There are twice $n(n-1)/2$ or $n(n-1) = n^2 - n$ off-diagonal entries.

Each of these $n(n-1)$ entries can can be a 0 or a 1.

So there are $2^{n(n-1)}$ reflexive adjacency matrices.
Counting Reflexive Relations

Let \( U \) be a set with cardinality \( n \).

A relation on \( U \) can be represented as a subset \( \sim \) of the Cartesian product \( U \times U \).

There are \( n \times n = n^2 \) ordered pairs in \( U \times U \).

To be reflexive, the subset \( \sim \) must contain \((u, u)\) for each \( u \in U \). That is, these \( n \) ordered pairs must be related.

There are \( n^2 - n = n(n - 1) \) other ordered pairs that can be in or out of \( \sim \).

So there are \( 2^{n(n-1)} \) subsets that represent reflexive relations.

Related Relation

- A relation \( \sim \) is not reflexive if there is a value \( x \) such that \( x \not\sim x \). In symbols,
  \[
  (\exists x \in U)(x \not\sim x)
  \]
  The relation "\( xy \) is even" on the natural numbers is not reflexive. For instance, \( 1 \cdot 1 \) is not even.

- A relation \( \sim \) is irreflexive if \( x \not\sim x \) for all \( x \). In symbols,
  \[
  (\forall x \in U)(x \not\sim x)
  \]
  Not equal and less than are both irreflexive.

For all values of \( x \), \( x \neq x \) and \( x \not\sim x \) both True.

Problems on Reflexive Relations

Show your understanding of this topic by completing the problems found at Reflexive Relations.