Representing Permutations

There are several ways to write a permutation.

As an example, let \( \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \).

A permutation on \( \mathbb{D} \) can be represented by

- The permutation itself, say \( \langle 0, 2, 4, 6, 8, 1, 3, 5, 7, 9 \rangle \), but this notation fails to show how the permutation is constructed.

- A \( 2 \times n \) permutation matrix

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & 9
\end{bmatrix}
\]

which shows how terms are permuted from their natural order.

- Cyclic notation

\[ [0][1, 5, 7, 8, 4, 2][3, 6][9] \]

Cyclic Notation for Permutations

Consider the 4-cycle permutation written in cyclic notation

\[ [0][1, 5, 7, 8, 4, 2][3, 6][9] \]

This notation is read

1. 0 goes to (position) 0.
2. 1 goes to (position) 5, 5 goes to 7, 7 goes to 8, 8 goes to 4, 4 goes to 2, and 2 goes to 1.
3. 3 goes to (position) 6 and 6 goes to 3.
4. 9 goes to (position) 9.

Therefore, the permutation is \( \langle 0, 2, 4, 6, 8, 1, 3, 5, 7, 9 \rangle \).

Writing a Permutation in Cyclic Notation

To write a permutation in cyclic notation it is perhaps best to first write it in matrix notation.

For instance, given the permutation \( \langle 8, 6, 4, 2, 0, 9, 7, 5, 3, 1 \rangle \) write it in matrix notation

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
8 & 6 & 4 & 2 & 0 & 9 & 7 & 5 & 3 & 1
\end{bmatrix}
\]

And now read off the cyclic structure.

(See the next slide)
Writing a Permutation in Cyclic Notation

Use the matrix notation

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
8 & 6 & 4 & 2 & 0 & 9 & 7 & 5 & 3 & 1
\end{bmatrix}
\]

to read off the cyclic structure.

1. 0 goes to (position) 4, 4 goes to 2, 2 goes to 3, 3 goes to 8, and 8 goes to 0. Giving the cycle

\[ [0, 4, 2, 3, 8] \]

2. 1 goes to 9, 9 goes to 5, 5 goes to 7, 7 goes to 6, and 6 goes to 1.

\[ [1, 9, 5, 7, 6] \]

Therefore, the permutation has 2 cycles \([0, 4, 2, 3, 8][1, 9, 5, 7, 6]\)

Writing a Permutation in Cyclic Notation

Here is another example.

Given the permutation \(\langle 7, 4, 8, 1, 3, 6, 5, 2, 9, 0 \rangle\) write it in matrix notation

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
7 & 4 & 8 & 1 & 3 & 6 & 5 & 2 & 9 & 0
\end{bmatrix}
\]

Read off the cyclic structure.

1. 0 goes to (position) 9, 9 goes to 8, 8 goes to 2, 2 goes to 7, and 7 goes to 0.

2. 1 goes to 3, 3 goes to 4, 4 goes to 1.

3. 5 goes to 6, 6 goes to 5.

Therefore, the permutation has 3 cycles

\[ [0, 9, 8, 2, 7][1, 3, 4][5, 6] \]

Writing Cyclic Notation as a Permutation

Consider the reverse problem:

Given cyclic notation, write the permutation.

For instance, given the four cycle permutation

\[ [0, 2, 5][3, 8, 7, 6][4, 1][9] \]

Write it in matrix notation

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
5 & 4 & 0 & 6 & 1 & 5 & 7 & 8 & 3 & 9
\end{bmatrix}
\]

and read off the permutation

\[ \langle 5, 4, 0, 6, 1, 5, 7, 8, 3, 9 \rangle \]
Permutations Written in Cyclic Notation

\begin{align*}
\begin{bmatrix} 0, 1, 2 \end{bmatrix} & \quad \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 1, 2 \end{bmatrix} \\
\begin{bmatrix} 0, 2, 1 \end{bmatrix} & \quad \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 0, 2 \end{bmatrix} \\
& \quad \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 0, 1 \end{bmatrix}
\end{align*}

Figure 1: Cyclic notation for the $3! = 6$ permutations of $\{0, 1, 2\}$.

The similar table and figure for $4! = 24$ permutations of $\{0, 1, 2, 3\}$ is shown.

Figure 2: Cyclic notation for the $4! = 24$ permutations of $\{0, 1, 2, 3\}$.