Functions that Solve Recurrence Equations

We will start simply. Warning! algebra ahead.

In an algebra class, you've learned how to show a number satisfies an equation.

For instance, you can show that $x = 2$ satisfies the equation

$$x^2 + 5x - 14 = 0$$

By substituting 2 for x and computing:

$$2^2 + 5 \cdot 2 - 14 = 4 + 10 - 14 = 0$$

Can you compute the other root?

Functions that Solve Recurrence Equations

To show that a function $F(n)$ solves a recurrence equation, you can substitute the formula for $F(n)$ into the equation, and verify the equality.

For instance, to show that $a(n) = 1$ solves

$$a_n = a_{n-1} \quad \text{or } a(n) = a(n-1) \text{ in function notation}$$

Complete the simple proof:

The Alice Equation. If $a(n) = 1$ for all n, then $a(n-1) = 1 = a(n)$. That is, the equation $a_n = a_{n-1}$ is satisfied by $a_n = a(n) = 1$. □

Do you agree every constant function $a(n) = c$ solves the equation $a_n = a_{n-1}$.?

The Gauss Recurrence Equation

A function that solves to the Gauss recurrence equation

$$g_n = g_{n-1} + 1$$

is $g(n) = g_n = n$ for $n \geq 0$.

The Gauss Equation. If $g(n) = n$ for all n, then $g(n) = n = (n-1) + 1 = g(n-1) + 1$. □

Do you agree every linear function $g(n) = n + c$ solves the equation $g_n = g_{n-1} + 1$.?

The Triangular Recurrence Equation

A function that solves to the triangular recurrence equation

$$t_n = t_{n-1} + (n-1)$$

is $t(n) = n(n-1)/2$ for all $n \geq 0$.

The Triangular Equation. If $t(n) = n(n-1)/2$ for all n, then

$$t(n) = \frac{n(n-1)}{2} = \frac{(n-2+2)(n-1)}{2}$$

$$t(n) = \frac{(n-2)(n-1)}{2} + \frac{2(n-1)}{2} = t(n-1) + (n-1)$$

□

The Mersenne Recurrence Equation

A function that solves to the Mersenne recurrence equation

$$m_n = 2m_{n-1} + 1$$

is $m(n) = 2^n - 1$

The Mersenne Equation. If $m(n) = 2^n - 1$ for all n, then

$$m(n-1) = 2^{n-1} - 1 \quad \text{and } 2m(n-1) + 1 = 2(2^{n-1} - 1) + 1$$

$$2m(n-1) + 1 == 2^n - 2 + 1 = m(n)$$

□

The Binary Search Recurrence Equation

A function that solves to the Binary search recurrence equation

$$b_n = b_{n/2} + 1$$

is $b(n) = \lg n$

The Binary Search Equation. If $b(n) = \lg n$ for all $n \geq 1$, then

$$b(n) = \lg n = \lg\left(2 \cdot \frac{n}{2}\right)$$

$$b(n) = \lg\left(\frac{n}{2}\right) + \lg 2 = b\left(\frac{n}{2}\right) + 1$$

□