Functions that Solve Recurrence Equations

We will start simply. **Warning!** algebra ahead.

In an algebra class, you’ve learned how to show a number satisfies an equation.

For instance, you can show that $x = 2$ satisfies the equation

$$x^2 + 5x - 14 = 0$$

by substituting $2$ for $x$ and computing:

$$2^2 + 5 \cdot 2 - 14 = 4 + 10 - 14 = 0$$

Can you compute the other root?

Functions that Solve Recurrence Equations

To show that a function $F(n)$ solves a recurrence equation, you can substitute the formula for $F(n)$ into the equation, and verify the equality.

For instance, to show that $a(n) = 1$ solves

$$a_n = a_{n-1}$$

or $a(n) = a(n - 1)$ in function notation

Complete the simple proof:

**The Alice Equation.** If $a(n) = 1$ for all $n$, then $a(n - 1) = 1 = a(n)$. That is, the equation $a_n = a_{n-1}$ is satisfied by $a_n = a(n) = 1$.

Do you agree every constant function $a(n) = c$ solves the equation $a_n = a_{n-1}$?

The Gauss Recurrence Equation

A function that solves the Gauss recurrence equation

$$g_n = g_{n-1} + 1$$

is $g(n) = g_n = n$ for $n \geq 0$.

**The Gauss Equation.** If $g(n) = n$ for all $n$, then $g(n) = n = (n - 1) + 1 = g(n - 1) + 1$.

Do you agree every linear function $g(n) = n + c$ solves the equation $g_n = g_{n-1} + 1$?
The Triangular Recurrence Equation
A function that solves to the triangular recurrence equation
\[ t_n = t_{n-1} + (n - 1) \]
is \( t(n) = n(n - 1)/2 \) for all \( n \geq 0 \).

The Triangular Equation. If \( t(n) = n(n - 1)/2 \) for all \( n \), then
\[
t(n) = \frac{n(n - 1)}{2} = \frac{(n - 2 + 2)(n - 1)}{2}
\]
\[
t(n) = \frac{(n - 2)(n - 1)}{2} + \frac{2(n - 1)}{2} = t(n - 1) + (n - 1)
\]

The Mersenne Recurrence Equation
A function that solves to the Mersenne recurrence equation
\[ m_n = 2m_{n-1} + 1 \]
is \( m(n) = 2^n - 1 \).

The Mersenne Equation. If \( m(n) = 2^n - 1 \) for all \( n \), then
\[
m(n - 1) = 2^{n-1} - 1 \quad \text{and} \quad 2m(n - 1) + 1 = 2(2^{n-1} - 1) + 1
\]
\[
2m(n - 1) + 1 = 2^n - 2 + 1 = m(n)
\]

The Binary Search Recurrence Equation
A function that solves to the binary search recurrence equation
\[ b_n = b_{n/2} + 1 \]
is \( b(n) = \lg n \).

The Binary Search Equation. If \( b(n) = \lg n \) for all \( n \geq 1 \), then
\[
b(n) = \lg n = \lg \left( 2 \cdot \frac{n}{2} \right)
\]
\[
b(n) = \lg \left( \frac{n}{2} \right) + \lg 2 = b \left( \frac{n}{2} \right) + 1
\]