The Symmetric Property of Relations

The symmetric property is defined by a conditional statement.

Let \( \sim \) be a relational symbol.

If \( x \sim y \) implies \( y \sim x \) for every \( x, y \in U \), then \( \sim \) is symmetric.

**Definition 1 (Symmetric Relation).** Let \( \sim \) be a relation on set \( U \). The relation \( \sim \) is **symmetric** if \( x \sim y \) implies \( y \sim x \) for all \( x, y \in U \).

In symbols, \( \sim \) is symmetric when

\[
(\forall x, y \in U)((x \sim y) \rightarrow (y \sim x))
\]

Representations of Symmetric Relations

If you recall, a relation is a set \( \sim \) of ordered pairs. **Symmetric** says if \( (x, y) \in \sim \), then \( (y, x) \in \sim \).

If you recall, a relation is an adjacency matrix, Symmetric says the value in row \( x \), column \( y \) equals the value in row \( y \), column \( x \).

\[
\begin{array}{cccccccc}
& a & b & c & d & \cdots & u \\
 a & x & \bullet & \bullet & \cdots & \bullet \\
b & \bullet & x & \bullet & 0 & \cdots & 0 \\
c & \bullet & \bullet & x & 1 & \cdots & 1 \\
d & \bullet & 0 & 1 & x & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u & \bullet & 0 & 1 & 0 & \cdots & x \\
\end{array}
\]

The diagonal values \( x \) do not matter.

Representations of Symmetric Relations

If you know about matrices, a relation is symmetric when its adjacency matrix equals its **transpose**. Row \( k \) of \( A \) is column \( k \) of \( A^T \).

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\( A^T \) is symmetric

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\( A^T \) is **not** symmetric
Examples of Symmetric Relations
The following relations are symmetric.

• Equality on the set of integers. \((\forall a, b \in \mathbb{Z})(\ (a = b) \rightarrow (b = a))\).

• Not equal on the set of integers. \((\forall a, b \in \mathbb{Z})(\ (a \neq b) \rightarrow (b \neq a))\).

• Congruence mod \(n\) on the set of integers. \((\forall a \in \mathbb{Z})(\ (a \equiv b \pmod{n}) \rightarrow (b \equiv a \pmod{n}))\).

  If \(a - b\) is a multiple of \(n\), then \(b - a\) is a multiple of \(n\).

Examples of Symmetric Relations
The following relations are symmetric.

• Points on an origin-centered circle: \((\forall (x, y) \in \mathbb{R}^2)(\ (x^2 + y^2 - 1 = 0) \rightarrow (y^2 + x^2 - 1 = 0))\).

![Graph](circle.png)

• Even products on the integers:

  If \(xy\) is even, then \(yx\) is even.

Examples of Relations that are Not Symmetric
The following relations are not symmetric.

• Less than on the set of integers. A counterexample is, \(1 < 2\) but \(2 \not< 1\).

• Divides on the set of natural numbers. A counterexample is, \(1 \mid 2\) but \(2 \not\mid 1\).

• Proper subset on \(2^\mathbb{N}\), the power set \(\mathbb{N}\). A counterexample is, \(\{0\} \subset \{0, 1\}\) but \(\{0, 1\} \not\subset \{0\}\).
Counting Symmetric Relations

It is interesting to count symmetric relations on a finite set.

**Theorem 2 (The Number of Symmetric Relations).** Let \( U \) be a set with cardinality \( n \). There are

\[
2^{n(n+1)/2} = \sqrt{2^n(n+1)}
\]

symmetric relations on \( U \).

Recall, a relation on \( U \) can be represented as an \( n \times n \) adjacency matrix \( A \).

There are \( n \times n = n^2 \) entries in \( A \), and each entries can take on one of 2 values.

There are \( 2^{n^2} \) different adjacency matrices (relations) that can be defined.

Using Adjacency Matrices to Count Symmetric Relations

Continuing from the previous slide.

To be symmetric, the matrix \( A \) must have equal values in symmetric entries: the value in row \( x \), column \( y \) equals the value in row \( y \), column \( x \).

Recall, there are

\[
1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}
\]

entries on or below the main diagonal of \( A \).

The value in an entry of \( A \) can either be True or False.

But, once a value for an off-diagonal entry is chosen the value of the symmetric entry is also determined.

There are

\[
2^{n(n+1)/2} = \sqrt{2^n(n+1)}
\]

ways to choose values on or below the main diagonal of \( A \), and for each of these choice you can construct a symmetric matrix.
Counting Symmetric Relations

Here’s another way to count symmetric relations. Let \( \mathbb{U} \) be a set with cardinality \( n \).

Let name and order the elements of \( \mathbb{U} \):
\[
\mathbb{U} = \{u_0, u_1, u_2, \ldots, u_{n-2}, u_{n-1}\}
\]
A relation on \( \mathbb{U} \) can be represented as a subset \( \sim \) of the Cartesian product \( \mathbb{U} \times \mathbb{U} \).

There are \( n \times n = n^2 \) ordered pairs in \( \mathbb{U} \times \mathbb{U} \).

To be symmetric, if \((x, y) \in \sim\) then \((y, x) \in \sim\).

Consider \( p(j, k) = u_j \sim u_k \) as a predicate statement. When creating a relation you can set \( p(j, k) \) to either True or False.

Counting Symmetric Relations

To make a symmetric relation you choose to set
\[
u_j \sim u_k \quad \text{to True or False}
\]
for \( j = 0, 1, \ldots, n-1 \) and \( k = j, j+1, \ldots, n-1 \). For a fixed \( j \), the index \( k \) steps through \( n-j \) decisions.

\[
(u_0, u_0) \in \sim \quad (u_0, u_1) \in \sim \quad (u_0, u_2) \in \sim \quad \cdots \quad (u_0, u_{n-1}) \in \sim \\
(u_1, u_1) \in \sim \quad (u_1, u_2) \in \sim \quad \cdots \quad (u_1, u_{n-1}) \in \sim \\
(u_2, u_2) \in \sim \quad \cdots \quad (u_2, u_{n-1}) \in \sim \\
\vdots \quad \vdots \\
(u_{n-1}, u_{n-1}) \in \sim
\]

There are \( 2^{n(n+1)/2} \) symmetric adjacency matrices.
Related Relation

• A relation $\sim$ is not symmetric when there are values $x$ and $y$ such that $x \sim y$ and $y \not\sim x$. In symbols,
  $$(\exists x, y \in U)((x \sim y) \land (y \not\sim x))$$

• A relation $\sim$ is asymmetric when $(x \sim y) \land (y \not\sim x)$, for all $x$ and $y$. In symbols,
  $$(\forall x, y \in U)((x \sim y) \rightarrow (y \not\sim x))$$

• A relation $\sim$ is antisymmetric when for all $x$ and $y$, if $x \neq y$, then $x \not\sim y$ or $y \not\sim x$. In symbols,
  $$(\forall x, y \in U)((x \neq y) \rightarrow ((x \not\sim y) \lor (y \not\sim x)))$$

Problems on Symmetric Relations

Show your understanding of this topic by completing the problems found at Symmetric Relations.