The Symmetric Property of Relations

The <u>symmetric property</u> is defined by a conditional statement.

Let $\sim$ be a relational symbol.

If $x \sim y$ implies $y \sim x$ for every $x, y \in \mathbb{U}$, then $\sim$ is <u>symmetric</u>.

Definition 1 (Symmetric Relation)**.** Let $\sim$ be a relation on set $\mathbb{U}$. The relation $\sim$ is symmetric if $x \sim y$ implies $y \sim x$ for all $x, y \in \mathbb{U}$.

In symbols, $\sim$ is symmetric when

$$(\forall x, y \in \mathbb{U})((x \sim y) \to (y \sim x))$$

Representations of Symmetric Relations

If you recall, a relation is a set $\sim$ of ordered pairs. <u>Symmetric</u> says if $(x, y) \in \sim$, then $(y, x) \in \sim$.

If you recall, a relation is an adjacency matrix, <u>Symmetric</u> says the value in row x, column y equals the value in row y, column x.

$$\begin{array}{c|cccccc}
 & a & b & c & d & \cdots & u \\
\hline
a & x & \bullet & \bullet & \bullet & \cdots & \bullet \\
b & \bullet & x & \bullet & 0 & \cdots & 0 \\
c & \bullet & \bullet & x & 1 & \cdots & 1 \\
d & \bullet & 0 & 1 & x & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
u & \bullet & 0 & 1 & 0 & \cdots & x
\end{array}$$

The diagonal values x do not matter.

Representations of Symmetric Relations

If you know about matrices, a relation is symmetric when its adjacency matrix equals its <u>transpose</u>. Row k of A is column k of A^T.

$$A = \begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 1 & 1 & 0 \\
b & 1 & 1 & 0 & 0 \\
c & 1 & 0 & 0 & 1 \\
d & 0 & 0 & 1 & 1
\end{array} = A^T \quad \text{is symmetric}$$

$$A = \begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 1 & 1 & 0 \\
b & 1 & 1 & 0 & 0 \\
c & 0 & 1 & 0 & 1 \\
d & 0 & 0 & 0 & 1
\end{array} \neq A^T \quad \text{is } \underline{\text{not}} \text{ symmetric}$$

Examples of Symmetric Relations

The following relations are symmetric.

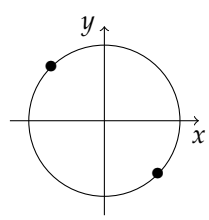
- Equality on the set of integers. $(\forall a, b \in \mathbb{Z})((a = b) \to (b = a))$.
- Not equal on the set of integers. $(\forall a, b \in \mathbb{Z})((a \neq b) \to (b \neq a))$.
- Congruence mod n on the set of integers. $(\forall a \in \mathbb{Z})((a \equiv b \pmod{n}) \to (b \equiv a \pmod{n}))$.

  If $a - b$ is a multiple of n, then $b - a$ is a multiple of n

Examples of Symmetric Relations

The following relations are symmetric.

- Points on an origin-centered circle: $(\forall (x, y) \in \mathbb{R}^2)((x^2 + y^2 - 1 = 0) \to (y^2 + x^2 - 1 = 0))$.



- Even products on the integers:

  If xy is even, then yx is even.

Examples of Relations that are Not Symmetric

The following relations are <u>not</u> symmetric.

- Less than on the set of integers. A counterexample is, $1 < 2$ but $2 \not< 1$.
- Divides on the set of natural numbers. A counterexample is, $1 \mid 2$ but $2 \nmid 1$.
- Proper subset on $2^{\mathbb{N}}$, the power set $\mathbb{N}$. A counterexample is, $\{0\} \subset \{0, 1\}$ but $\{0, 1\} \not\subset \{0\}$.

Counting Symmetric Relations

It is interesting to count symmetric relations on a finite set.

Theorem 2 (The Number of Symmetric Relations)**.** *Let $\mathbb{U}$ be a set with cardinality n. There are*

$$2^{n(n+1)/2} = \sqrt{2^{n(n+1)}}$$

symmetric relations on $\mathbb{U}$.

Recall, a relation on $\mathbb{U}$ can be represented as an $n \times n$ adjacency matrix A.

There are $n \times n = n^2$ entries in A, and each entries can take on one of 2 values.

There are 2^{n^2} different adjacency matrices (relations) that can be defined.

Using Adjacency Matrices to Count Symmetric Relations

Continuing from the previous slide.

To be symmetric, the matrix A must have equal values in symmetric entries: the value in row x, column y equals the value in row y, column x.

Recall, there are

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

entries on or below the main diagonal of A.

The value in an entry of A can either be `True` or `False`.

But, once a value for an off-diagonal entry is chosen the value of the symmetric entry is also determined.

There are

$$2^{n(n+1)/2} = \sqrt{2^{n(n+1)}}$$

ways to choose values on or below the main diagonal of A, and for each of these choice you can construct a symmetric matrix.

Counting Symmetric Relations

Here's another way to count symmetric relations. Let $\mathbb{U}$ be a set with cardinality n.

Let name and order the elements of $\mathbb{U}$.

$$\mathbb{U} = \{u_0, u_1, u_2, \ldots, u_{n-2}, u_{n-1}\}$$

A relation on $\mathbb{U}$ can be represented as a subset $\sim$ of the Cartesian product $\mathbb{U} \times \mathbb{U}$.

There are $n \times n = n^2$ ordered pairs in $\mathbb{U} \times \mathbb{U}$.

To be symmetric, if $(x, y) \in \sim$ then $(y, x) \in \sim$.

Consider $\mathtt{p}(j, k) = u_j \sim u_k$ as a predicate statement. When creating a relation you can set $\mathtt{p}(j, k)$ to either `True` or `False`.

Counting Symmetric Relations

To make a symmetric relation you choose to set

$$u_j \sim u_k \quad \text{to } \texttt{True} \text{ or } \texttt{False}$$

for $j = 0, 1, \ldots, n-1$ and $k = j, j+1, \ldots, n-1$. For a fixed j, the index k steps through $n - j$ decisions.

$$\begin{array}{ccccc}
(u_0, u_0) \in \sim & (u_0, u_1) \in \sim & (u_0, u_2) \in \sim & \cdots & (u_0, u_{n-1}) \in \sim \\
 & (u_1, u_1) \in \sim & (u_1, u_2) \in \sim & \cdots & (u_1, u_{n-1}) \in \sim \\
 & & (u_2, u_2) \in \sim & \cdots & (u_2, u_{n-1}) \in \sim \\
 & & & \vdots & \vdots \\
 & & & & (u_{n-1}, u_{n-1}) \in \sim
\end{array}$$

There are $2^{n(n+1)/2}$ symmetric adjacency matrices.

Related Relation

- A relation $\sim$ is <u>not symmetric</u> when there are values x and y such that $x \sim y$ and $y \not\sim x$. In symbols,

$$(\exists x, y \in \mathbb{U})((x \sim y) \wedge (y \not\sim x))$$

- A relation $\sim$ is <u>asymmetric</u> when $(x \sim y) \wedge (y \sim x)$, for all x and y. In symbols,

$$(\forall x, y \in \mathbb{U})((x \sim y) \rightarrow (y \not\sim x))$$

- A relation $\sim$ is <u>antisymmetric</u> when for all x and y, if $x \neq y$, then $x \not\sim y$ or $y \not\sim x$. In symbols,

$$(\forall x, y \in \mathbb{U})((x \neq y) \rightarrow ((x \not\sim y) \vee (y \not\sim x))$$

Problems on Symmetric Relations

Show your understanding of this topic by completing the problems found at Symmetric Relations.