Truth Assignments to Boolean Variables

It is useful to know in how many ways True and False can be assigned to a set of Boolean variables.

For one variable $p$, there are two truth assignments

\[ p = \text{False} = 0 \quad \text{or} \quad p = \text{True} = 1 \]

Truth Assignments to Boolean Variables

For two variables $p$ and $q$, there are four truth assignments

\[
\begin{align*}
& p = \text{False} = 0, \quad q = \text{False} = 0 \\
& p = \text{False} = 0, \quad q = \text{True} = 1 \\
& p = \text{True} = 1, \quad q = \text{False} = 0 \\
& p = \text{True} = 1, \quad q = \text{True} = 1
\end{align*}
\]

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For three variables $p$, $q$, and $r$, there are eight truth assignments. It is convenient to arrange these truth assignments into a table.

\[
\begin{array}{ccc}
\text{p} & \text{q} & \text{r} \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

Do you see the pattern used to construct the table?
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For four variables \( p, q, r, \) and \( s \), there are sixteen truth assignments. The truth assignment table for these four variables has 16 rows. Form it by duplicating the table for 3 variables appending a 0 in each row of one copy and a 1 in each row of the second copy.

\[
\begin{array}{cccc}
  p & q & r & s \\
  \hline
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 1 \\
  0 & 1 & 0 & 0 \\
  0 & 1 & 0 & 1 \\
  0 & 1 & 1 & 0 \\
  0 & 1 & 1 & 1 \\
  p & q & r & s \\
  \hline
  1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 1 \\
  1 & 0 & 1 & 0 \\
  1 & 0 & 1 & 1 \\
  1 & 1 & 0 & 0 \\
  1 & 1 & 0 & 1 \\
  1 & 1 & 1 & 0 \\
  1 & 1 & 1 & 1 \\
\end{array}
\]

Counting Truth Assignments

Given \( n \) Boolean variables, how many different truth assignments can be made?

<table>
<thead>
<tr>
<th>Number of Variables</th>
<th>Number of Truth Assignments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 1 = 2^0 )</td>
</tr>
<tr>
<td>1</td>
<td>( 2 = 2^1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 4 = 2^2 )</td>
</tr>
<tr>
<td>3</td>
<td>( 8 = 2^3 )</td>
</tr>
<tr>
<td>4</td>
<td>( 16 = 2^4 )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( n )</td>
<td>( 2^n )</td>
</tr>
</tbody>
</table>

Truth Assignments to Boolean Variables

**Theorem 1.** There are \( 2^n \) truth assignments on \( n \) Boolean variables.

**Proof.**

1. The statement is True for \( n = 1 \).

2. Pretend the statement is True for some \( n \geq 1 \). That is, there are \( 2^n \) truth assignments on \( n \) Boolean variables.

3. Then, for an \((n + 1)^{\text{th}}\) variable there are \( 2^n \) assignments where it is False and \( 2^n \) assignments where it is True. Therefore, there are

\[
2^n + 2^n = 2^{n+1}
\]

truth assignments on \( n + 1 \) variables.

\[ \square \]