CSE 1400 Applied Discrete Mathematics
Sequences, Induction & Recursion

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Problems labeled

⚠ are important and must be mastered.

are harder and cover interesting advanced material that is not properly part of this course.

are tangential to the course but cover material that is useful elsewhere.

Problems on Sequence

Background

Sequences are objects on which induction and recursion are commonly applied. A sequence, written

$$\vec{S} = \langle s_0, s_1, s_2, \ldots \rangle$$

is an ordered list of terms s_k, $k = 0, 1, 2, \ldots$.

Induction is used to demonstrate that if some predicate $\mathrm{P}(s_{n-1})$ is `True`, then $\mathrm{P}(s_n)$ is also `True`, which with a basis that $\mathrm{P}(s_0)$ is `True` we can conclude that $(\forall n \in \mathbb{N})(\mathrm{P}(s_n))$ is `True`.

$$[\mathrm{P}(s_0) \wedge (\mathrm{P}(s_{n-1}) \rightarrow \mathrm{P}(s_n))] \rightarrow (\forall n \in \mathbb{N})(\mathrm{P}(s_n))$$

In a stronger form, to derive that $\mathrm{P}(s_n)$ is `True`, induction requires several or all predicates $\mathrm{P}(s_k)$ to be `True` for $k < n$.

Recursion defines s_n as a function of s_{n-1} or more lower order terms.

1. ⚠ (Know basic concepts about sequences.) `True` or `False`: A sequence $\langle a_0, a_1, \ldots, a_{n-1}, \ldots \rangle$ is an ordered list of objects, called terms.

   Answer: This is `True`. A sequence can be written $\langle a_0, a_1, \ldots, a_{n-1}, \ldots \rangle$. The sequence is ordered in that the term at position (index) $n-1$ occurs before the term at index n.

2. ⚠ (Know basic concepts about sequences.) `True` or `False`: A sequence can contain the same term multiple times.

   Answer: This is `True`: $\langle 0, 1, 1, 2, 3, 5, \ldots \rangle$ is the Fibonacci sequence and it contains 1 twice.

3. (Know basic concepts about sequences and lists.) `True` or `False`: A list is a sequence of finite length.

   Answer: The term "list" used for a finite sequence. Most of the sequences we will discuss can be continued indefinitely, but in computing practice finite sequence, lists, are used. For example, $\langle 0, 1, 1, 2, 3, 5, 8 \rangle$ is a list of length 7. The notation $|\vec{S}|$ is used to denote the length of a list.

4. (Know basic concepts about sequences.) `True` or `False`: $\langle\rangle$ is the empty sequence with no terms.

   Answer: This is the notation I would like to use for the empty sequence.

5. ⚠ (Know some fundamental sequences.) Below are commonly occurring sequences. Describe these sequences.

(a) The Alice sequence $\vec{A}$

Answer: $\vec{A}$ is the "Alice" sequence

$$\vec{A} = \langle 1, 1, 1, \ldots, 1, \ldots \rangle$$

The Alice sequence is the first column in Pascal's triangle. The terms in the Alice sequence are the binomial coefficients

$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$

which is the number of ways to choose no objects from a set of n objects.

(b) The Gauss sequence $\vec{G}$

Answer: $\vec{G}$ is the Gauss sequence

$$\vec{G} = \langle 0, 1, 2, 3, 4, 5, 6, 7, \ldots, (n-1), \ldots \rangle$$

The terms in the Gauss sequence are the binomial coefficients

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$$

which is the number of ways to choose 1 object from a set of n objects for $n \in \mathbb{N}$. The Gauss sequence can be computed as sums of terms in the Alice sequence.

$$n = g_n = \sum_{k=0}^{n-1} a_k = \sum_{k=0}^{n-1} 1$$

(c) The Triangular sequence $\vec{T}$

Answer: $\vec{T}$ is the Triangular sequence

$$\vec{T} = \left\langle 0, 0, 1, 3, 6, 10, 15, 21, \ldots, \frac{n(n-1)}{2}, \ldots \right\rangle$$

The terms in the Triangular sequence are the binomial coefficients

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

which is the number of ways to choose 2 objects from a set of n objects for $n \in \mathbb{N}$. The Triangular sequence can be computed as sums of terms in the Gauss sequence.

$$\frac{n(n-1)}{2} n = t_n = \sum_{k=0}^{n-1} g_k = \sum_{k=0}^{n-1} k$$

(d) The power of 10 sequence $\vec{10}$

Answer: $\vec{10}$ is the power of 10 sequence.

$$\vec{10} = \langle 1, 10, 100, 1000, 10000, \ldots, 10^n, \ldots \rangle$$

The powers of 10 are the standard basis for writing numbers (decimal notation):

$$31.415 = 3 \times 10^1 + 10^0 + 4 \times 10^{-1} + 10^{-2} + 5 \times 10^{-3}$$

(e) The power of 2 sequence $\vec{2}$
Answer: $\vec{2}$ is the power of 2 sequence.

$$\vec{2} = \langle 1, 2, 4, 8, 16, \ldots, 2^n, \ldots \rangle$$

The powers of 2 are the binary basis for writing numbers (binary notation):

$$14.4375 = 2^3 + 2^2 + 2^1 + 2^{-2} + 2^{-3} + 2^{-4}$$

(f) The Mersenne sequence $\vec{M}$
Answer: $\vec{M}$ is the Mersenne sequence

$$\langle M \rangle = \langle 0, 1, 3, 7, 15, 31, 63, 127, \ldots, (2^n - 1), \ldots \rangle$$

The Mersenne sequence can be computed as sums of terms in the power of 2 sequence.

$$2^n - 1 = m_n = \sum_{k=0}^{n-1} 2^k$$

(g) The Fibonacci sequence $\vec{F}$
Answer: $\vec{F}$ is the Fibonacci sequence

$$\vec{F} = \langle 0, 1, 1, 2, 3, 5, 8, 13, \ldots, (f_{n-1} + f_{n-2}), \ldots \rangle$$

(h) The Harmonic sequence $\vec{H}$
Answer: $\vec{H}$ is the Harmonic sequence

$$\vec{H} = \left\langle 0, 1, 3/2, 11/6, 50/24, 274/120, \ldots, (H_{n-1} + \frac{1}{n}), \ldots \right\rangle$$

(i) The Busy Beaver sequence $\vec{B}$
Answer: $\vec{B}$ is the Busy Beaver sequence

$$\vec{B} = \langle 1, 4, 6, 13, ? \rangle$$

Terms in the sequence are the maximum number of 1's that a deterministic, n state (with one additional halting state), two-way infinite initially blank tape, Turing machine can write using 1 as the only non-blank tape symbol. The sequence cannot be computed. Only the first four terms have been determined.

6. (Know functions that enumerate some fundamental sequences.) Match the sequence with the function that maps the natural numbers to terms in the sequence.

(a) (e) $\vec{A}$
(b) (b) $\vec{G}$
(c) (c) $\vec{T}$
(d) (a) $\vec{M}$
(e) (d) $\vec{F}$

(a) $f(n) = 2^n - 1$
(b) $f(n) = n$
(c) $t(n) = n(n-1)/2$
(d) $f(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$
(e) $f(n) = 1$

7. ⚠ (Know recurrence equations and initial conditions that enumerate some fundamental sequences.) Match the sequence with the recurrence equation and initial condition(s) that enumerate the terms in the sequence.

(a) (e) $\vec{A}$
(b) (a) $\vec{G}$
(c) (d) $\vec{T}$
(d) (f) $\vec{M}$
(e) (b) $\vec{F}$
(f) (c) $\vec{H}$

(a) $f_n = f_{n-1} + 1$, $n \geq 1$, $f_0 = 0$
(b) $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$, $f_0 = 0$, $f_1 = 1$
(c) $f_n = f_{n-1} + 1/n$, $n \geq 1$, $f_0 = 0$
(d) $f_n = f_{n-1} + (n-1)$, $n \geq 1$, $f_0 = 0$
(e) $f_n = f_{n-1}$, $n \geq 1$, $f_0 = 1$
(f) $f_n = 2f_n + 1$, $n \geq 1$, $f_0 = 0$

8. ⚠ (Be able to show a function solves a recurrence equation.) Show the function solves the recurrence equation.

(a) Function: $a(n) = 1$, Recurrence equation: $a_n = a_{n-1}$.
Answer: If $a(n) = 1$ then $a(n-1) = 1$ and $a(n) = a(n-1)$.

(b) Function: $g(n) = n$, Recurrence equation: $g_n = g_{n-1} + 1$.
Answer: If $g(n) = n$ then $g(n-1) = n-1$ and $g(n) = g(n-1) + 1$.

(c) Function: $t(n) = n(n-1)/2$, Recurrence equation: $t_n = t_{n-1} + (n-1)$.
Answer: If $t(n) = n(n-1)/2$ then $t(n-1) = (n-1)(n-2)/2$ and $t(n) = t(n-1) + (n-1)$, that is

$$\begin{aligned}\frac{(n-1)(n-2)}{2} + (n-1) &= \frac{(n-1)(n-2)}{2} + \frac{2(n-1)}{2} \\ &= \frac{(n-1)}{2}(n-2+2) \\ &= \frac{n(n-1)}{2}\end{aligned}$$

(d) Function: $m(n) = 2^n - 1$, Recurrence equation: $m_n = 2m_{n-1} + 1$.
Answer: If $m(n) = 2^n - 1$ then $m(n-1) = 2^{n-1} - 1$ and $2^n - 1 = 2(2^{n-1} - 1) + 1$.

9. (Be able to determine a function that computes the partial sums of a sequence.) Let $\vec{S}_n = \langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle$ be a list of length n. The sum function $\sum$ folds terms in list into their sum

$$\text{sum}(\vec{S}_n) = \sum_{k=0}^{n-1} s_k = s_0 + s_1 + s_2 + \cdots + s_{n-1}$$

(a) What is the function that computes the following sums? (All sums are from $k = 0$ to $k = n-1$)

i. $\text{sum}(\vec{A}_n) = \sum 1$?
Answer: The sum of the first n Alice numbers is n.

ii. $\text{sum}(\vec{G}_n) = \sum k$?
Answer: The sum of the first n Gauss numbers is $n(n-1)/2$.

iii. $\text{sum}(\vec{T}_n) = \sum k(k-1)/2$?
Answer: The sum of the first n Triangular numbers is $n(n-1)(n-2)/6$.

iv. $\text{sum}(\vec{2}_n) = \sum 2^k$?
Answer: The sum of the first n powers of 2 is $2^n - 1$.

v. $\text{sum}(\vec{10}_n) = \sum 10^k$?
Answer: The sum of the first n powers of 10 is $(10^n - 1)/9$.

vi. $\text{sum}(\vec{F}_n) = \sum f_k$?
Answer: The sum of the first n Fibonacci numbers is $f_{n+1} - 1$.

(b) `True` or `False`: The sum function satisfies the recurrence equation

$$\text{sum}(\vec{S}_n) = \text{sum}(\vec{S}_{n-1}) + s_{n-1}$$

Explain your answer.
Answer: `True`. Explanation: The sum of the first n terms of the sequence $\vec{S}$ is equal to the sum of the first $n-1$ terms of $\vec{S}$ plus s_n, the nth term.

10. (Be able to determine a function that computes the partial products of a sequence.) Let $\vec{S}_n = \langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle$ be a list of length n. The product function $\prod$ folds terms in the sequence into their product

$$\text{prod}(\vec{S}_n) = \prod_{k=0}^{n-1} s_k = s_0 \cdot s_1 \cdot s_2 \cdots s_{n-1}$$

(a) What is the function that computes the following products?

i. $\text{prod}(\vec{A}_n) = \prod_{k=0}^{n-1} 1$?
Answer: The product of the first n Alice numbers is 1.

ii. $\prod_{k=1}^{n} k$
Answer: The product of the first n non-zero Gauss numbers is $n!$.

iii. $\prod_{k=2}^{n+1} k(k-1)/2$?
Answer: The product of the first n non-zero Triangular numbers is $(n+1)!n!/2^n$.

iv. $\prod_{k=0}^{n-1} 2^k$?
Answer: The product of the first n powers of 2 is $2^{n(n-1)/2}$.

v. $\prod_{k=0}^{n-1} 10^k$?
Answer: The product of the first n powers of 10 is $10^{n(n-1)/2}$.

(b) `True` or `False`: The product function satisfies the recurrence equation

$$\text{prod}(\vec{S}_n) = s_{n-1}\text{prod}(\vec{S}_{n-1})$$

Answer: `True`. Explanation: The product of the first n terms of the sequence $\vec{S}$ is equal to the product of s_{n-1}, the nth term, and the product of the first $n-1$ terms of $\vec{S}$.

11. (Know some operations that are applied to lists.) List are fundamental types for computing. Programming languages often have functions that operate on lists. How would you define the following?

(a) $|\vec{S}|$
Answer: $|\vec{S}|$ returns the length of a list. For example $|\langle 0, 1, 3, 7, 15 \rangle| = 5$.

(b) $\text{head}(\vec{S})$
Answer: $\text{head}(\vec{S})$ returns the first element in the non-empty list $\vec{S}$. For example $\text{head}\langle 0, 1, 3, 7, 15 \rangle = 0$.

(c) $\text{tail}(\vec{S})$
Answer: $\text{tail}(\vec{S})$ returns the list after the head of the the non-empty list $\vec{S}$. For example $\text{tail}\langle 0, 1, 3, 7, 15\rangle = \langle 1, 3, 7, 15\rangle$.

(d) $\vec{R} + \vec{S}$
Answer: $\vec{R} + \vec{S}$ concatenates list $\vec{R}$ and $\vec{S}$. For example,$\langle 0, 1, 3, 7\rangle + \langle 0, 1, 1, 2\rangle = \langle 0, 1, 3, 7, 0, 1, 1, 2\rangle$.

(e) $\text{map}(f, \vec{S})$ where f is a function on the natural numbers.
Answer: $\text{map}(f, \vec{S})$ creates the list

$$\langle f(s_0), f(s_1), f(s_2), f(s_3), \ldots\rangle$$

where

$$\vec{S} = \langle s_0, s_1, s_2, s_3, \ldots\rangle$$

12. (Understand the idea of a subsequence.) A subsequence $\vec{T}$ of sequence $\vec{S}$ is a sequence

$$\vec{T} = \langle t_0, t_1, t_2, \ldots, t_{n-1}, \ldots\rangle$$

such that each term in $\vec{T}$ is also a term in $\vec{S}$, that is, $(\forall k \in \mathbb{N})(\exists i \in \mathbb{N})(t_k = s_i)$, and the terms in $\vec{T}$ are ordered as those in $\vec{S}$.

(a) `True` or `False`: The even subsequence of the natural numbers is $\langle 0, 2, 4, \ldots\rangle = \langle 2k : k \in \mathbb{N}\rangle$.

(b) `True` or `False`: The odd subsequence of the natural numbers is $\langle 1, 3, 5, \ldots\rangle = \langle 2k + 1 : k \in \mathbb{N}\rangle$.

(c) `True` or `False`: The power of two subsequence of the natural numbers is $\langle 1, 2, 4, \ldots\rangle = \left\langle 2^k : k \in \mathbb{N}\right\rangle$.

13. (Know some number theoretic functions and their sequences.) The divisor function $d(n)$, $n \geq 1$ counts the number of divisors of n. Mapping the divisor function from the sequence of positive integers produces the sequence

$$\vec{\mathcal{D}} = \langle d_1, d_2, d_3, d_4, d_5, d_6, d_7, \ldots\rangle = \langle 1, 2, 2, 3, 2, 4, 2, \ldots\rangle$$

(a) What is $d(6)$, $d(8)$, $d(9)$ and $d(12)$?
Answer: $d(6) = 4$ (6 has divisors 1, 2, 3, and 6), $d(8) = 4$ (8 has divisors 1, 2, 4, and 8), $d(9)$ (9 has divisors 1, 3, and 9), and $d(12) = 6$ (12 has divisors 1, 2, 3, 4, 6 and 12)

(b) What does $d(n) = \sum_{d|n} d^0$ mean?
Answer: Since $d^0 = 1$ the sum adds a 1 each time the number d divides n, that is, the sum $\sum_{d|n} d^0$ counts the number of divisors of n.

(c) `True` or `False`: The divisor function is multiplicative, that is $d(nm) = d(n)d(m)$ when $\gcd(n, m) = 1$.
Answer: This is `True`. As an example, $6 = d(12) = d(3)d(4) = 2 \cdot 3$. The proof applies the fundamental theorem of arithmetic.

14. (Know some number theoretic functions and their sequences.) The (Euler) totient function $\varphi(n)$, $n \geq 1$ counts the number of positive integers less than n that are relatively prime to n. Mapping the totient function from the sequence of positive integers produces the sequence

$$\vec{\tau} = \langle \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8, \ldots\rangle = \langle 1, 1, 2, 2, 4, 2, 6, 4, \ldots\rangle$$

(a) What is $\varphi(6)$?
Answer: $\varphi(6) = 2$ because only 1 and 5 are relatively prime to and less than 6.

(b) What is $\varphi(8)$?
Answer: $\varphi(8) = 4$ because only 1, 3, 5, and 7 are relatively prime to and less than 8.

(c) What is $\varphi(10)$?
Answer: $\varphi(10) = 4$ because only 1, 3, 7, and 9 are relatively prime to and less than 10.

(d) What is $\varphi(7)$?
Answer: $\varphi(5) = 4$ because 1, 2, 3, and 4 are relatively prime to and less than 5.

(e) What is $\varphi(7)$?
Answer: $\varphi(7) = 6$ because 1, 2, 3, 4, 5, and 6 are relatively prime to and less than 7.

(f) What is $\varphi(p)$ when p is prime?
Answer: $\varphi(p) = p - 1$ because each natural number $1, 2, 3, \ldots, (p-1)$ are relatively prime to and less than p.

(g) Show that for $n = 6, 8, 10$

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where p is a prime divisor of n. (The formula is correct for all n.)
Answer:

$$\begin{aligned}
\varphi(6) &= 6 \prod_{p|6} \left(1 - \frac{1}{p}\right) \\
&= 6 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \\
&= 6 \frac{1}{2} \frac{2}{3} = 2 \\
\varphi(8) &= 8 \prod_{p|8} \left(1 - \frac{1}{p}\right) \\
&= 8 \left(1 - \frac{1}{2}\right) \\
&= 4 \\
\varphi(10) &= 10 \prod_{p|10} \left(1 - \frac{1}{p}\right) \\
&= 10 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{5}\right) \\
&= 10 \left(\frac{1}{2}\right) \left(\frac{4}{5}\right) = 4
\end{aligned}$$

Problems on Induction

Background

Mathematical induction is one of the most important proof techniques in discrete mathematics because the recurrences they establish lead directly to computational solutions.

1. (Understand and be able to use summation notation.) Consider the sum

$$a_0 + a_1 + a_2 + \cdots + a_{n-1} = \sum_{k=0}^{n-1} a_k$$

(a) When $n = 0$ the sum is said to be empty. What is the value of the empty sum

$$\sum_{k=0}^{-1} a_k$$

Answer: The empty sum has value 0. It is the sum of no terms.

(b) What is the value of the sum when $n = 4$ and $a_k = k$?
Answer: The value of the sum is $0 + 1 + 2 + 3 = 6$.

(c) What is the value of the sum when $n = 4$ and $a_k = 2^k$?
Answer: The value of the sum is $1 + 2 + 4 + 8 = 15$.

2. (Understand and be able to use product notation.) Consider the product

$$a_0 a_1 a_2 \cdots a_{n-1} = \prod_{k=0}^{n-1} a_k$$

(a) When $n = 0$ the product is said to be empty. What is the value of the empty product

$$\prod_{k=0}^{-1} a_k$$

Answer: The empty product has value 1. It is the product of no factors.

(b) What is the value of the product when $n = 4$ and $a_k = k + 1$?
Answer: The value of the product is $1 \cdot 2 \cdot 3 \cdot 4 = 24$.

(c) What is the value of the product when $n = 4$ and $a_k = 2^k$?
Answer: The value of the product is $1 \cdot 2 \cdot 4 \cdot 8 = 2^{0+1+2+3} = 2^6$.

3. (Understand the basic concept of induction.) Consider a predicate $\mathrm{P}(n)$ on the natural numbers. Suppose that $\mathrm{P}(0)$ is `True` and suppose that if $\mathrm{P}(n-1)$ is `True`, then $\mathrm{P}(n)$ is `True`, that is,

$$\mathrm{P}(0) \wedge (\mathrm{P}(n-1) \rightarrow \mathrm{P}(n))$$

is `True`. What conclusion can you draw?

Answer: $\mathrm{P}(n)$ is `True` for all natural numbers:

$$\begin{array}{c} \mathrm{P}(0) \text{ is } \texttt{True} \\ \mathrm{P}(0) \rightarrow \mathrm{P}(1) \text{ is } \texttt{True}, \text{ so } \mathrm{P}(1) \text{ is } \texttt{True} \\ \mathrm{P}(1) \rightarrow \mathrm{P}(2) \text{ is } \texttt{True}, \text{ so } \mathrm{P}(2) \text{ is } \texttt{True} \\ \vdots \quad \vdots \end{array}$$

4. ⚠ (Be able to prove the sums of natural numbers are Triangle numbers.) Use mathematical induction to prove the sum of the first n natural numbers is $n(n-1)/2$. That is,

$$\sum_{0 \le k < n} k = \frac{n(n-1)}{2}$$

Answer: The natural numbers lie in the set

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$$

The sum of the first n natural numbers is

$$0 + 1 + 2 + 3 + \cdots + (n-1)$$

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality is empty and has value zero. Also, the right-hand side expression is $0(0-1)/2$ is also zero.

Inductive Premise: Pretend the equality

$$\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}$$

is `True` for some $n \ge 0$.

Inductive Step: If

$$\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}$$

then

$$\begin{aligned}
\sum_{k=0}^{n} k &= \sum_{k=0}^{n-1} k + n \\
&= \frac{n(n-1)}{2} + n \\
&= \frac{n(n-1)}{2} + \frac{2n}{2} \\
&= \frac{n(n-1) + 2n}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

Conclusion:

$$\sum_{k=0}^{n} k = \frac{n(n-1)}{2} \quad (\forall n \in \mathbb{N})$$

5. (Be able to prove variations of problem 3.) Use mathematical induction to prove the following.

(a) The sum of the even natural numbers from 0 to $2n$ is $n(n+1)$. That is,

$$\sum_{k=0}^{n} 2k = n(n+1)$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality has value zero. Also, the right-hand side expression is zero.

Inductive Premise: The equality

$$\sum_{k=0}^{n} 2k = n(n+1)$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n} 2k = n(n+1)$$

then

$$\begin{aligned}\sum_{k=0}^{n+1} 2k &= \sum_{k=0}^{n} 2k + (2n+2) \\ &= n(n+1) + 2(n+1) \\ &= (n+1)(n+2)\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

Conclusion:

$$\sum_{k=0}^{n} 2k = n(n+1) \quad (\forall n \in \mathbb{N})$$

(b) The sum of the odd natural numbers from 1 to $2n+1$ is $(n+1)^2$. That is,

$$\sum_{k=0}^{n} (2k+1) = (n+1)^2$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality has value 1. Also, the right-hand side expression is 1.

Inductive Premise: The equality

$$\sum_{k=0}^{n} (2k+1) = (n+1)^2$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n} (2k+1) = (n+1)^2$$

then

$$\begin{aligned}\sum_{k=0}^{n+1} (2k+1) &= \sum_{k=0}^{n} (2k+1) + (2n+3) \\ &= (n+1)^2 + (2n+3) \\ &= n^2 + 4n + 4 \\ &(n+2)^2\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

Conclusion:

$$\sum_{k=0}^{n}(2k+1) = (n+1)^2 \quad (\forall n \in \mathbb{N})$$

(c) The sum of the natural numbers that have a remainder of 2 when divided by 3 from 2 to $3n+2$ is $(n+1)(3n+4)/2$. That is,

$$\sum_{k=0}^{n}(3k+2) = \frac{(n+1)(3n+4)}{2}$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality has value 2. Also, the right-hand side expression is $(0+1)(3 \cdot 0+4)/2 = 2$.

Inductive Premise: The equality

$$\sum_{k=0}^{n}(3k+2) = \frac{(n+1)(3n+4)}{2}$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n}(3k+2) = \frac{(n+1)(3n+4)}{2}$$

then

$$\begin{aligned}\sum_{k=0}^{n+1}(3k+2) &= \sum_{k=0}^{n}(3k+2) + (3n+5) \\ &= \frac{(n+1)(3n+4)}{2} + (3n+5) \\ &= \frac{3n^2+7n+4+6n+10}{2} \\ &\frac{(n+2)(3n+7)}{2}\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

6. ⚠ (Be able to prove the sum powers of 2 are Mersenne numbers.) Use mathematical induction to prove the sum of the first n powers of 2 is $2^n - 1$. That is,

$$\sum_{0 \leq k < n} 2^k = 2^n - 1$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality is empty and has value zero. Also, the right-hand side expression $2^0 - 1$ is zero also.

Inductive Premise: Pretend the equality

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1$$

then

$$\begin{aligned}\sum_{k=0}^{n} 2^k &= \sum_{k=0}^{n-1} 2^k + 2^n \\ &= (2^n - 1) + 2^n \\ &= 2 \cdot 2^n - 1 \\ &= 2^{n+1} - 1\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

Conclusion:

$$\sum_{k=0}^{n} 2^k = 2^n - 1 \quad (\forall n \in \mathbb{N})$$

7. (Recognize the relationship of sums of consecutive pairs of natural numbers to Pascal's triangle.) Use mathematical induction to prove the sum of products consecutive pairs of natural numbers is the product of three consecutive number divided by 3, that is,

$$\sum_{k=0}^{n-1} k(k-1) = \frac{n(n-1)(n-2)}{3}$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality is empty and has value zero. Also, the right-hand side expression is zero.

Inductive Premise: The equality

$$\sum_{k=0}^{n-1} k(k-1) = \frac{n(n-1)(n-2)}{3}$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n-1} k(k-1) = \frac{n(n-1)(n-2)}{3}$$

then

$$\begin{aligned}\sum_{k=0}^{n} k(k-1) &= \sum_{k=0}^{n-1} k(k-1) + n(n-1) \\ &= \frac{n(n-1)(n-2)}{3} + n(n-1) \\ &= \frac{n(n-1)(n-2) + 3n(n-1)}{3} \\ &= \frac{n(n-1)[(n-2)+3]}{3} \\ &= \frac{(n+1)n(n-1)}{3}\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

Conclusion:

$$\sum_{k=0}^{n-1} k(k-1) = \frac{n(n-1)(n-2)}{3} \quad (\forall n \in \mathbb{N})$$

8. (Be able to manipulate algebraic fractions.) Use mathematical induction to prove the summation formula

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

is true for all natural numbers $n \geq 0$.

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality is empty and has value zero. Also, the right-hand side expression is zero for $n = 0$.

Inductive Premise: Pretend the equality

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

then

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{(2k-1)(2k+1)} &= \sum_{k=0}^{n} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2n+1)(2n+3)} \\
&= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\
&= \frac{n(2n+3)+1}{(2n+1)(2n+3)} \\
&= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\
&= \frac{(n+1)}{2n+3}
\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

Conclusion:

$$\sum_{k=0}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1} \quad (\forall n \in \mathbb{N})$$

9. (Be able to manipulate algebraic expressions.) Use mathematical induction to prove the sum the cubes is a square, that is,

$$\sum_{k=0}^{n-1} k^3 = \left(\frac{n(n-1)}{2}\right)^2$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality is empty and has value zero. Also, the right-hand side expression is zero for $n = 0$.

Inductive Premise: Pretend the equality

$$\sum_{k=0}^{n-1} k^3 = \left(\frac{n(n-1)}{2}\right)^2$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n-1} k^3 = \left(\frac{n(n-1)}{2}\right)^2$$

then

$$\begin{aligned}
\sum_{k=0}^{n} k^3 &= \sum_{k=0}^{n-1} k^3 + n^3 \\
&= \left(\frac{n(n-1)}{2}\right)^2 + n^3 \\
&= \left(\frac{n(n-1)}{2}\right)^2 + \frac{4n^3}{4} \\
&= \left(\frac{n^2(n-1)^2 + 4n^3}{4}\right) \\
&= n^2\left(\frac{(n-1)^2 + 4n}{4}\right) \\
&= n^2\left(\frac{(n+1)^2}{4}\right) \\
&= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n + 1$.

10. (Be able to manipulate factorial expressions.) Use mathematical induction to prove the sum of k times $k!$ from $k = 0$ to $k = n - 1$ is $n! - 1$

$$\sum_{k=0}^{n-1} k \cdot k! = n! - 1.$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality is empty and has value zero. Also, the right-hand side expression is zero.

Inductive Premise: Pretend the equality

$$\sum_{k=0}^{n-1} k \cdot k! = n! - 1$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n-1} k \cdot k! = n! - 1$$

then

$$\begin{aligned}\sum_{k=0}^{n} k \cdot k! &= \sum_{k=0}^{n-1} k \cdot k! + n \cdot n! \\ &= (n! - 1) + n \cdot n! \\ &= n!(n+1) - 1 \\ &= (n+1)! - 1\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

11. (Be able to manipulate factorial expressions.) Use mathematical induction to prove the summation formula

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$$

is `True` whenever n is a natural number greater than 1.

Answer:

Basis for Induction: For $n = 1$ the sum on the left-hand side of the equality is 1. Also, the right-hand side expression is $2 - 1/1 = 1$.

Inductive Premise: Pretend the equality

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$$

is `True` for some $n \geq 1$.

Inductive Step: If

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$$

then

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{k^2} &= \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \\ &< 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\ &= 2 - \left(\frac{(n+1)^2 - n}{n(n+1)^2}\right) \\ &= 2 - \frac{n^2 + n + 1}{n(n+1)^2} \\ &= 2 - \frac{n(n+1)}{n(n+1)^2} - \frac{1}{n(n+1)^2} \\ &= 2 - \frac{1}{n+1} - \frac{1}{n(n+1)^2} \\ &< 2 - \frac{1}{n+1}\end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

12. ⚠ (Understand the divides relation.) Prove 6 divides $7^n - 1$ for all natural numbers n.
Answer: Well, 6 divides $7^0 - 1 = 0$ and $7^1 - 1 = 6$ establishing a basis for induction. And if 6 divides $7^{n-1} - 1$ then 6 divides

$$\begin{aligned} 7^n - 1 &= 7^n - 7^{n-1} + 7^{n-1} - 1 \\ &= 7^{n-1}(7-1) + (7^{n-1} - 1) \end{aligned}$$

13. (Know the growth rate of factorials versus exponentials.) Prove $n! \geq 2^n$ for all natural numbers $n \geq 4$.
Answer: $4! = 24 \geq 2^4 = 16$. And, if $n! \geq 2^n$, then $(n+1)! = (n+1)n! \geq (n+1)2^n > 2 \cdot 2^n = 2^{n+1}$.

14. (Know relationships about Fibonacci numbers.) Let F_n denote a term in the Fibonacci sequence

n	0	1	2	3	4	5	6	7	8	9	...
F_n	0	1	1	2	3	5	8	13	21	34	...

(a) Use mathematical induction to show that the sum of the first n Fibonacci numbers is a Fibonacci number minus one. Specifically, show that

$$F_0 + F_1 + \cdots + F_{n-1} = \sum_{k=0}^{n-1} = F_{n+1} - 1$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side of the equality is empty and has value 0. On the right-hand side of the equality, for $n = 0$, $F_{0+1} - 1 = F_1 - 1 = 0$. This establishes the basis is true.

Inductive Premise: Pretend the equality

$$\sum_{k=0}^{n-1} F_k = F_{n+1} - 1$$

is `True` for some $n \geq 0$.

Inductive Step: If

$$\sum_{k=0}^{n-1} F_k = F_{n+1} - 1$$

then

$$\begin{aligned} \sum_{k=0}^{n} F_k &= \sum_{k=0}^{n-1} F_k + F_n \\ &= (F_{n+1} - 1) + F_n \\ &= (F_{n+1} + F_n) - 1 \\ &= F_{n+2} - 1 \end{aligned}$$

establishing that if the equality holds for n, then it holds for $n+1$.

Conclusion:

$$\sum_{k=0}^{n-1} F_k = F_{n+1} \quad (\forall n \in \mathbb{N})$$

(b) Use mathematical induction to prove that

$$F_n \leq 2F_{n-1} \quad \text{for } n \geq 2.$$

Answer:

Basis for Induction: For $n = 2$, $1 = F_2 \leq 2F_1 = 2$, which establishes the basis.

Induction: For some $n \geq 2$, pretend that $F_n \leq 2F_{n-1}$. Then

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\leq F_n + F_n \\ &= 2F_n \end{aligned}$$

(c) Use induction over k to prove that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

Answer:

Basis for Induction: For $k = 1$, $F_{n+1} = F_1 F_{n+1} + F_0 F_n$ since $F_1 = 1$ and $F_0 = 0$ establishing the basis.

Note you could use the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with $n = 1$ to deduce that $1 = F_1 = F_0 + F_{-1} = 0 + F_{-1}$ so that $F_{-1} = 1$. Therefore, for $k = 0$, $F_{n+0} = F_0 F_{n+1} + F_{-1} F_n = 0 \cdot F_{n+1} + 1 \cdot F_n = F_n$.

Induction: For some $k \geq 1$, pretend that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

and

$$F_{n+k-1} = F_{k-1} F_{n+1} + F_{k-2} F_n$$

Then

$$\begin{aligned} F_{n+k+1} &= F_{n+k} + F_{n+k-1} \\ &= (F_k F_{n+1} + F_{k-1} F_n) + (F_{k-1} F_{n+1} + F_{k-2} F_n) \\ &= (F_k + F_{k-1}) F_{n+1} + (F_{k-1} + F_{k-2}) F_n \\ &= F_{k+1} F_{n+1} + F_k F_n \end{aligned}$$

(d) (Know the greatest common divisor function.) Prove that for $n \geq 1$

$$\gcd(F_n, F_{n-1}) = 1$$

Answer:

Basis for Induction: For $n = 1$, $\gcd(F_1, F_0) = \gcd(1, 0) = 1$ since 1 is the largest natural number that divides both 1 and 0.

Induction: Pretend that

$$\gcd(F_n, F_{n-1}) = 1$$

is `True` for some $n \geq 1$. Then

$$\begin{aligned}\gcd(F_{n+1}, F_n) &= \gcd(F_n + F_{n-1}, F_n) \\ &= \gcd(F_{n-1}, F_n) \\ &= 1\end{aligned}$$

(e) Prove that the sum of the first $n - 1$ Fibonacci numbers is 1 less than the $n + 1$ Fibonacci number. That is,

$$F_n = 1 + \sum_{k=0}^{n-2} F_k \quad \text{if } n \geq 2$$

Answer:

Basis for Induction: For $n = 2$, $F_2 = 1$ and $1 + F_0 = 1$ establishing the basis.

Induction: For some $n \geq 2$, pretend that

$$F_n = 1 + \sum_{k=0}^{n-2} F_k$$

Then

$$\begin{aligned}1 + \sum_{k=0}^{n-1} F_k &= 1 + \sum_{k=0}^{n-2} F_k + F_{n-1} \\ &= F_n + F_{n-1} \\ &= F_{n+1}\end{aligned}$$

(f) Prove that the sum of the odd-indexed Fibonacci numbers up to F_{2n+1} equals F_{2n+2}. That is,

$$\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}$$

Answer:

Basis for Induction: For $n = 0$,

$$\sum_{k=0}^{0} F_{2k+1} = F_1 = 1$$

and $F_{2\cdot 0+2} = F_2 = 1$ establishing the basis.

Induction: Pretend that

$$\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}$$

is `True` for some $n \geq 0$. Then

$$\begin{aligned}\sum_{k=0}^{n+1} F_{2k+1} &= \sum_{k=0}^{n} F_{2k+1} + F_{2n+3} \\ &= F_{2n+2} + F_{2n+3} \\ &= F_{2n+4}\end{aligned}$$

(g) Prove that

$$\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$$

Answer:

Basis for Induction: For $n = 0$,

$$\sum_{k=0}^{0} F_{2k} = F_0 = 0$$

and $F_{2\cdot 0+1} - 1 = F_1 - 1 = 0$ establishing the basis.

Induction: Pretend that

$$\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$$

is `True` for some $n \geq 0$. Then

$$\begin{aligned}\sum_{k=0}^{n+1} F_{2k} &= \sum_{k=0}^{n} F_{2k} + F_{2n+2} \\ &= (F_{2n+1} - 1) + F_{2n+2} \\ &= (F_{2n+1} + F_{2n+2}) - 1 \\ &= F_{2n+3} - 1\end{aligned}$$

(h) (Know relationships between Fibonacci number and binomial coefficients.) Prove that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{0}{n} = \sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

Answer: Note that $\binom{n}{k} = 0$ when $n < k$.

Basis for Induction: For $n = 0$,

$$\sum_{k=0}^{0} \binom{0-k}{k} = \binom{0}{0} = 1$$

and $F_{0+1} = F_1 = 1$. Since we'll use Pascal's identity a second basis case must be established. For $n = 1$,

$$\sum_{k=0}^{1} \binom{1-k}{k} = \binom{1}{0} + \binom{0}{1} = 1 + 0 = 1$$

and $F_{1+1} = F_2 = 1$.

Induction: Pretend that

$$\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{0}{n-1} = \sum_{k=0}^{n-} \binom{n-1-k}{k} = F_n$$

and

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{0}{n} = \sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

are `True` for some $n \geq 1$. Then

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1-k}{k} &= \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n+1-k}{k} + \binom{0}{n+1} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n+1-k}{k} + \binom{0}{n+1} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \left(\binom{n-k}{k-1} + \binom{n-k}{k} \right) \\
&= \left(\sum_{k=1}^{n} \binom{n-k}{k-1} \right) + \left(\sum_{k=0}^{n} \binom{n-k}{k} \right) \\
&= \left(\sum_{k=0}^{n-1} \binom{n-1-k}{k} \right) + \left(\sum_{k=0}^{n} \binom{n-k}{k} \right) \\
&= F_n + F_{n+1} \\
&= F_{n+2}
\end{aligned}$$

(i) Prove Cassini's identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \quad \text{for } n > 0$$

where F_k is a Fibonacci number.
Answer: Start by replacing F_{n-1} by $F_{n+1} - F_n$ on the left-hand side of Cassini's identity to obtain

$$\begin{aligned}
F_{n+1}F_{n-1} - F_n^2 &= F_{n+1}(F_{n+1} - F_n) - F_n^2 \\
&= F_{n+1}^2 - F_{n+1}F_n - F_n^2 \\
&= F_{n+1}^2 - F_n(F_{n+1} + F_n) \\
&= F_{n+1}^2 - F_nF_{n+2} \\
&= -(F_nF_{n+2} - F_{n+1}^2) \\
&= -(F_{n+2}F_n - F_{n+1}^2)
\end{aligned}$$

Thus, if

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

Then

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$$

And since $F_2F_0 - F_1 = 1 \cdot 0 - 1 = (-1)^1$, the basis for induction is `True`, and this establishes Cassini's identity for all natural numbers n.

(j) Define the Lucas numbers L_n by

$$L_n = L_{n-1} + L_{n-2}, \text{ for } n > 1, \quad L_0 = 2,\ L_1 = 1$$

i. Show that the Lucas numbers L_n satisfy the equation

$$L_n = F_{n+1} + F_{n-1}$$

Answer: Recall from problem 14c that $F_{-1} = 1$.

Basis for Induction: For $n = 0$ we have $L_0 = 2$ and $F_{0+1} + F_{0-1} = F_1 + F_{-1} = 1 + 1 = 2$. Also, $L_1 = 1$ and $F_{1+1} + F_{1-1} = F_2 + F_0 = 1 + 0 = 1$. This establishes the basis.

Induction: Pretend that

$$L_n = F_{n+1} + F_{n-1} \quad \text{and} \quad L_{n+1} = F_{n+2} + F_n$$

are `True` for some $n \geq 0$ Then

$$\begin{aligned} L_{n+2} &= L_{n+1} + L_n \\ &= (F_{n+2} + F_n) + (F_{n+1} + F_{n-1}) \\ &= (F_{n+2} + F_{n+1}) + (F_n + F_{n-1}) \\ &= F_{n+3} + F_{n+1} \end{aligned}$$

ii. (Know relationships between Fibonacci and Lucas numbers.) Prove that

$$F_{2n} = F_n L_n$$

where F_n is a Fibonacci number.

Answer: Using the result

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

from problem 14c with $k = n$ to get

$$\begin{aligned} F_{2n} &= F_n F_{n+1} + F_{n-1} F_n \\ &= F_n (F_{n+1} + F_{n-1}) \\ &= F_n L_n \end{aligned}$$

15. (Know the relationship between columns in Pascal's rectangle.) Prove that the sum of terms in a column of Pascal's triangle equals a term in the next column, that is,

$$\sum_{k=m}^{n-1} \binom{k}{m} = \binom{n}{m+1}, \quad n \geq m+1$$

Answer:

Basis for Induction: For $n = m + 1$ the sum on the left-hand side is $\binom{m}{m} = 1$ and the right-hand side is $\binom{m+1}{m+1} = 1$ establishing the basis.

Induction: Pretend that

$$\sum_{k=m}^{n-1} \binom{k}{m} = \binom{n}{m+1}$$

is `True` for some $n \geq m+1$. Then

$$\begin{aligned}\sum_{k=m}^{n} \binom{k}{m} &= \sum_{k=m}^{n-1} \binom{k}{m} + \binom{n}{m} \\ &= \binom{n}{m+1} + \binom{n}{m} \\ &= \binom{n+1}{m+1} \quad \text{by Pascal's identity}\end{aligned}$$

Notice the pattern in

Binomial Coefficients $\binom{n}{m}$

		Choose m									
		0	1	2	3	4	5	6	7	8	9
	0	1									
	1	1	1								
	2	1	2	1							
	3	1	3	3	1						
n	4	1	4	6	4	1					
	5	1	5	10	10	5	1				
	6	1	6	15	20	15	6	1			
	7	1	7	21	35	35	21	7	1		
	8	1	8	28	56	70	56	28	8	1	
	9	1	9	36	84	126	126	84	36	9	1

16. ⚠ (Know the sum of terms in a row of Pascal's triangle is a power of 2.) Prove that the sum of term in a row of Pascal's triangle equals a power of 2, that is,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Answer:

Basis for Induction: For $n = 0$ the sum on the left-hand side is $\binom{0}{0} = 1$ and the right-hand side is $2^0 = 1$ establishing the basis.

Induction: Pretend that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

is `True` for some $n \geq 0$. Then

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n+1}{k} + \binom{n+1}{n+1} \\
&= \left(\binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k-1} \right) + \left(\sum_{k=1}^{n} \binom{n}{k}) + \binom{n}{0} \right) \\
&= 2^n + 2^n \\
&= 2^{n+1}
\end{aligned}$$

17. (Be able to manipulate expressions involving Harmonic numbers.) Use mathematical induction to prove that the Harmonic solve the recurrence

$$\sum_{k=1}^{n-1} H_k + n = nH_n$$

Answer:

Basis for Induction: For $n = 0$, the left-hand side summation is empty and thus equal to 0. For $n = 0$, the right-hand expression is equal to 0 also.

Inductive Premise: Pretend that

$$\sum_{k=1}^{n-1} H_k + n = nH_n$$

is `True` for some $n \geq 0$.

Inductive Step: Using the premise

$$\begin{aligned}
\sum_{k=1}^{n} H_k + (n+1) &= \left[\sum_{k=1}^{n-1} H_k + H_n \right] + (n+1) \\
&= \left[\sum_{k=1}^{n-1} H_k + n \right] + H_n + 1 \\
&= nH_n + H_n + 1 \\
&= (n+1)H_n + 1 \\
&= (n+1)(H_{n+1} - \frac{1}{n+1}) + 1 \\
&= (n+1)H_{n+1}
\end{aligned}$$

Problems on Recurrences

Background

Recurrence equations with initial value(s) are discrete analogs of differential equations studied in continuous mathematics. Two (or more) dimensional recursions with boundary conditions are discrete analogs of partial differential equations.

You will learn how to compute solutions to (some)

- First-order recurrence equations of the form $s_n = f(s_{n-1})$ where $f(x)$ is a simple function, for example

$$s_n = 2s_{n-1} + 1$$

- Second-order recurrence equations of the form $s_n = f(s_{n-1}, s_{n-2})$ where $f(x)$ is a simple function, for example
$$s_n = s_{n-1} + s_{n-2}$$
- Two-dimensional recurrence equations of the form $s_{n,k} = f(s_{n-1,k}, s_{n-1,k-1})$ where $f(x)$ is a simple function, for example
$$s_{n,k} = s_{n-1,k} + s_{n-1,k-1}$$

In the one dimensional case, initial conditions prove a basis for recursively computing all values, while in two-dimensions boundary conditions serve this purpose.

1. (Know the initial conditions and recurrence equations for some useful sequences.) Find initial conditions and recurrence equations that generate the following sequences.
   (a) The Alice sequence $\vec{A} = \langle 1, 1, 1, 1, \ldots 1 \rangle$
   Answer: Initial condition: $a_0 = 1$, Recurrence equation $a_n = a_{n-1}$.
   (b) The Gauss sequence $\vec{G} = \langle 0, 1, 2, 3, \ldots, (n-1) \rangle$
   Answer: Initial condition: $g_0 = 0$, Recurrence equation $g_n = g_{n-1} + 1$.
   (c) The Triangular sequence $\vec{T} = \langle 0, 0, 1, 3, 6, 10, \ldots, n(n-1)/2 \rangle$
   Answer: Initial condition: $t_0 = 0$, Recurrence equation $t_n = t_{n-1} + (n-1)$.
   (d) The Doubling sequence $\vec{D} = \langle 1, 2, 4, 8, 16, \ldots, 2^{n-1} \rangle$
   Answer: Initial condition: $d_0 = 1$, Recurrence equation $d_n = 2d_{n-1}$.
   (e) The Mersenne sequence $\vec{M} = \langle 0, 1, 3, 7, 15, \ldots, 2^{n-1} - 1 \rangle$
   Answer: Initial condition: $m_0 = 0$, Recurrence equation $m_n = 2m_{n-1} + 1$.

2. (Know the boundary conditions and recurrence equations for some useful sequences.) Find boundary conditions and recurrence equations that generate the following sequences.
   (a) The Binomial (Pascal) sequence

$$\vec{B} = \begin{array}{llllllll} \langle 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ 1 & 3 & 3 & 1 & & & & \\ 1 & 4 & 6 & 4 & 1 & & & \\ 1 & 5 & 10 & 10 & 5 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \rangle \end{array}$$

   Answer: Boundary conditions: $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$. Recurrence equation
$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

(b) The Stirling sequence of the first kind

$$\vec{S}_1 = \begin{array}{llllllll} \langle 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 1 & 1 & & & & & \\ 0 & 2 & 3 & 1 & & & & \\ 0 & 6 & 11 & 6 & 1 & & & \\ 0 & 24 & 50 & 35 & 10 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \rangle \end{array}$$

Answer: Boundary conditions: $\left[{n \atop 0}\right] = 0$, $n \neq 0$ and $\left[{n \atop n}\right] = 1$. Recurrence equation

$$\begin{bmatrix} n \\ m \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$$

(c) The Stirling sequence of the second kind

$$\vec{S}_2 = \begin{array}{llllllll} \langle 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 1 & 1 & & & & & \\ 0 & 1 & 3 & 1 & & & & \\ 0 & 1 & 7 & 6 & 1 & & & \\ 0 & 1 & 15 & 25 & 10 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \rangle \end{array}$$

Answer: Boundary conditions: $\left\{{n \atop 0}\right\} = 0$, $n \neq 0$ and $\left\{{n \atop n}\right\} = 1$. Recurrence equation

$$\begin{Bmatrix} n \\ m \end{Bmatrix} = m\begin{Bmatrix} n-1 \\ m \end{Bmatrix} + \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}$$

3. (Find a function and initial conditions that satisfy a recurrence.) Find a recurrence equation and initial condition that enumerates the sequence of even natural numbers $\langle 0, 2, 4, 8, \ldots \rangle$.

   Answer: The recurrence equation $e_n = e_{n-1} + 2$ for all $n \geq 1$ with initial condition $e_0 = 0$ will enumerate the even natural numbers.

4. (Find a function and initial conditions that satisfy a recurrence.) What change(s) must be made to your answer to problem 3 to enumerate the sequence of odd natural numbers?

   Answer: Change the initial condition to $e_0 = 1$.

5. (Know how to show a function satisfies a recurrence equation.) Prove the function $T(n) = 3^n - 2$ satisfies the recurrence equation

$$T_n = 3T_{n-1} + 4$$

   with initial condition $T_0 = -1$

Answer: Note that $T(0) = 3^0 - 2 = -1 = T_0$. Also note that, under the usual substitution,

$$\begin{aligned} 3T_{n-1} + 4 &= 3(3^{n-1} - 2) + 4 \\ &= 3^n - 2 \\ &= T_n \end{aligned}$$

6. (Know the recurrence for the common logarithm.) Use mathematical induction to prove that the common logarithm function $T_n = \log(n)$ satisfies the recurrence equation $T_{10n} = T_n + 1$ with initial condition $T_1 = 0$.
   Answer: Since $\log 1 = 0$ the initial condition is satisfied. Since $\log 10n = \log 10 + \log n = 1 + \log n$ the recurrence equation is satisfied.

7. ⚠ (Know the recurrence for the base 2 logarithm.) Prove the function $T(n) = \lg(n)$ satisfies the recurrence equation

$$T_{2n} = T_n + 1$$

   with initial condition $T_1 = 0$.
   Answer: Note that $T(1) = \lg(1) = 0 = T_1$. Also note that, under the usual substitution,

$$\begin{aligned} T_n + 1 &= \lg(n) + 1 \\ &= \lg(n) + \lg(2) \\ &= \lg(2n) \\ &= T_{2n} \end{aligned}$$

8. ⚠ (Know the recurrence for $n \lg n$.) Prove the function $T(n) = n \lg(n)$ satisfies the recurrence equation

$$T_{2n} = 2T_n + 2n$$

   with initial condition $T_1 = 0$.
   Answer: Note that $T(1) = 1 \lg(1) = 0 = T_1$. Also note that, under the usual substitution,

$$\begin{aligned} 2T_n + 2n &= 2n \lg(n) + 2n \\ &= 2n(\lg n + 1) \\ &= 2n(\lg n + \lg 2) \\ &= 2n \lg 2n \\ &= T_{2n} \end{aligned}$$

9. (Know how to show a function satisfies a (second-order) recurrence equation.) Use mathematical induction to prove that $s_n = 2^n + 3^n$ solves $s_n = 5s_{n-1} - 6s_{n-2}$ with initial conditions $s_0 = 2$, $s_1 = 5$.

   Answer:

   Basis for Induction: For $n = 0$, $s_0 = 2^0 + 3^0 = 2$ matches the initial conditions. For $n = 1$, $s_1 = 2^1 + 3^1 = 5$ matches the initial conditions.

   Induction Premise: For some $n \geq 2$, pretend that $s_{n-1} = 2^{n-1} + 3^{n-1}$ and $s_{n-2} = 2^{n-2} + 3^{n-2}$ correctly compute terms in the sequence.

   Inductive Step: Using the recurrence and the premise, we have

$$\begin{aligned}
s_n &= 5s_{n-1} - 6s_{n-2} \\
&= 5(2^{n-1} + 3^{n-1}) - 6(2^{n-2} + 3^{n-2}) \\
&= [5(2^{n-1}) - 6(2^{n-2})] + [5(3^{n-1}) - 6(3^{n-2})] \\
&= [5(2^{n-1}) - 3(2^{n-1})] + [5(3^{n-1}) - 2(3^{n-1})] \\
&= [2(2^{n-1})] + [3(3^{n-1})] \\
&= 2^n + 3^n
\end{aligned}$$

10. (Know how to show a function satisfies a (first or second-order) recurrence equation.) Use mathematical induction to prove the following recurrence equations and initial conditions are solved by the given functions.

    (a) $t_n = 3t_{n-1}$, $t_0 = 2$, $t(n) = 2 \cdot 3^n$.

    Answer:

    Basis for Induction: For $n = 0$ the function $t(n) = 2 \cdot 3^n$ computes $t_0 = 2$.

    Inductive Premise: For some $n \geq 0$, pretend the function $t(n-1) = 2 \cdot 3^{n-1}$ computes t_{n-1}. Then

$$\begin{aligned}
t_n &= 3t_{n-1} \\
&= 3(2 \cdot 3^{n-1}) \\
&= 2 \cdot 3^n
\end{aligned}$$

    (b) $t_n = 3t_{n-1} + 2$, $t_0 = 0$, $t(n) = 3^n - 1$.

    Answer:

    Basis for Induction: For $n = 0$ the function $t(n) = 3^n - 1$ computes $t_0 = 0$.

    Inductive Premise: For some $n \geq 0$, pretend the function $t(n-1) = 3^{n-1} - 1$ computes t_{n-1}. Then

$$\begin{aligned}
t_n &= 3t_{n-1} + 2 \\
&= 3(3^{n-1} - 1) + 2 \\
&= 3^n - 3 + 2 \\
&= 3^n - 1
\end{aligned}$$

    (c) $t_n = 5t_{n-1} - 6t_{n-2}$, $t_0 = 2$, $t_1 = 5$, $t(n) = 2^n + 3^n$.

    Answer:

Basis for Induction: For $n = 0$ the function $t(n) = 2^n + 3^n$ computes $t_0 = 2$, and for $n = 1$ the function $t(n) = 2^n + 3^n$ computes $t_1 = 5$

Inductive Premise: For some $n \geq 2$, pretend the function $t(n-2) = 2^{n-2} + 3^{n-2}$ computes t_{n-2}, and $t(n-1) = 2^{n-1} + 3^{n-1}$ computes t_{n-1}. Then

$$\begin{aligned} t_n &= 5t_{n-1} - 6t_{n-2} \\ &= 5(2^{n-1} + 3^{n-1}) - 6(2^{n-2} + 3^{n-2}) \\ &= 5 \cdot 2^{n-1} + 5 \cdot 3^{n-1} - 3 \cdot 2^{n-1} - 2 \cdot 3^{n-1} \\ &= 2 \cdot 2^{n-1} + 3 \cdot 3^{n-1} \\ &= 2^n + 3^n \end{aligned}$$

(d) $t_n = 2t_{n-1} + t_{n-2}$, $t_0 = 2$, $t_1 = 2$, $t(n) = (1+\sqrt{2})^n + (1-\sqrt{2})^n$.

Answer: Notice that $1+\sqrt{2}$ and $1-\sqrt{2}$ are solutions to the quadratic equation $x^2 - 2x - 1 = 0$. In particular, $2(1+\sqrt{2}) + 1 = (1+\sqrt{2})^2$ and $2(1-\sqrt{2}) + 1 = (1-\sqrt{2})^2$.

Basis: For $n = 0$, $t(0) = 2$, and for $n = 1$, $t(1) = 2$.

Hypothesis: Assume that $t(n-1) = t_{n-1}$ and $t(n-2) = t_{n-2}$ for some $n \geq 2$.

Step: Then

$$\begin{aligned} t_n &= 2t_{n-1} + t_{n-2} \\ &= 2\left[(1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n-1}\right] + \left[(1+\sqrt{2})^{n-2} + (1-\sqrt{2})^{n-2}\right] \\ &= (1+\sqrt{2})^{n-2}(2(1+sqrt2)+1) + (1-\sqrt{2})^{n-2}(2(1-\sqrt{2})+1) \\ &= (1+\sqrt{2})^{n-2}(1+sqrt2)^2 + (1-\sqrt{2})^{n-2}(1-\sqrt{2})^2 \\ &= (1+\sqrt{2})^n + (1-\sqrt{2})^n \end{aligned}$$

11. (Know when a sequence is arithmetic.) Let $\vec{A} = \langle a_0, a_1, a_2, \ldots, a_n, \ldots \rangle$ be an arithmetic sequence. That is, terms in $\vec{A}$ are given by a linear function $a(n) = mn + b$ for some slope m and y-intercept b. Find a recurrence equation and initial condition that enumerates the terms in the sequence.

Answer: $a_n = a_{n-1} + m$ is a recurrence that computes a value in the sequence from the value of the previous term, and $a_0 = b$ is the initial condition.

12. (Know when a sequence is geometric.) Let $\vec{\mathcal{G}} = \langle g_0, g_1, g_2, \ldots, g_n, \ldots \rangle$ be a geometric sequence. That is, terms in $\vec{G}$ are given by a exponential function $g(n) = br^n$ for some ratio r and y-intercept b. Find a recurrence equation and initial condition that enumerates the terms in the sequence.

Answer: $g_n = rg_{n-1}$ is a recurrence that computes a value in the sequence from the value of the previous term, and $g_0 = b$ is the initial condition.

13. (Know the Fermat numbers and their recurrence) Use mathematical induction to prove that the function $r(n) = 2^{2^n} + 1$ solves the Fermat recurrence

$$r_n = r_{n-1}^2 - 2(r_{n-2} - 1)^2$$

with initial conditions $r_0 = 3$ and $r_1 = 5$.

Answer: For the basis, $r(0) = 2^{2^0} + 1 = 2^1 + 1 = 3$ matching the first initial condition. Because the recurrence is second-order we need a second initial condition, and in this case $r(1) == 2^{2^1} + 1 =$

$2^2 + 1 = 5$, matching the second initial condition. Now let's pretend that $r(n-1) = 2^{2^{n-1}} + 1 = r_{n-1}$ and $r(n-2) = 2^{2^{n-2}} + 1 = r_{n-2}$. Then

$$\begin{aligned}
r_{n-1}^2 - 2(r_{n-2} - 1)^2 &= \left(2^{2^{n-1}} + 1\right)^2 - 2\left(2^{2^{n-2}}\right)^2 \\
&= 2^{2^{n-1}} 2^{2^{n-1}} + 2 \cdot 2^{2^{n-1}} + 1 - 2 \cdot 2^{2^{n-2}} 2^{2^{n-2}} \\
&= 2^{2^{n-1}+2^{n-1}} + 2^{2^{n-1}+1} + 1 - 2 \cdot 2^{2^{n-2}+2^{n-2}} \\
&= 2^{2^{n-1}+2^{n-1}} + 2^{2^{n-1}+1} + 1 - 2 \cdot 2^{2^{n-1}} \\
&= 2^{2^n} + 2^{2^{n-1}+1} + 1 - 2^{2^{n-1}+1} \\
&= 2^{2^n} + 1 \\
&= r(n)
\end{aligned}$$

14. (Be able to construct a recurrence equation by examining initial cases.) In how many ways can $n \geq 1$ be expressed as a sum of 1's and 2's? Different term orders are considered different sums.

Answer: Let $s(n)$ be the number of different ways to write n as a sum of 1's and 2's. Clearly $s(1) = 1$, $s(2) = 2$, and $s(3) = 3$

$$\begin{aligned}
1 &= 1 \\
2 &= 1 + 1 \\
2 &= 2 \\
3 &= 2 + 1 \\
3 &= 1 + 1 + 1 \\
3 &= 1 + 2
\end{aligned}$$

To compute $s(n)$ notice that n can be written as 2 plus any of the ways $n - 2$ can be written as a sum of 1's and 2's or n can be written as 1 plus any of the ways $n - 1$ can be written as a sum of 1's and 2's That is, $s(n) = s(n-1) + s(n-2)$, for example,

$$\begin{aligned}
4 &= 2 + 1 + 1 \\
4 &= 2 + 2 \\
4 &= 1 + 2 + 1 \\
4 &= 1 + 1 + 1 + 1 \\
4 &= 1 + 1 + 2
\end{aligned}$$

so $s(4) = 5$.

15. (Be able to construct a recurrence equation by examining initial cases.) Morse code is a series of dots ($\cdot$) and dashes (—). Pretend a dot can be typed in 1 second while a dash takes 2 seconds to type. Find a recurrence that describes how many messages can be typed in n seconds.

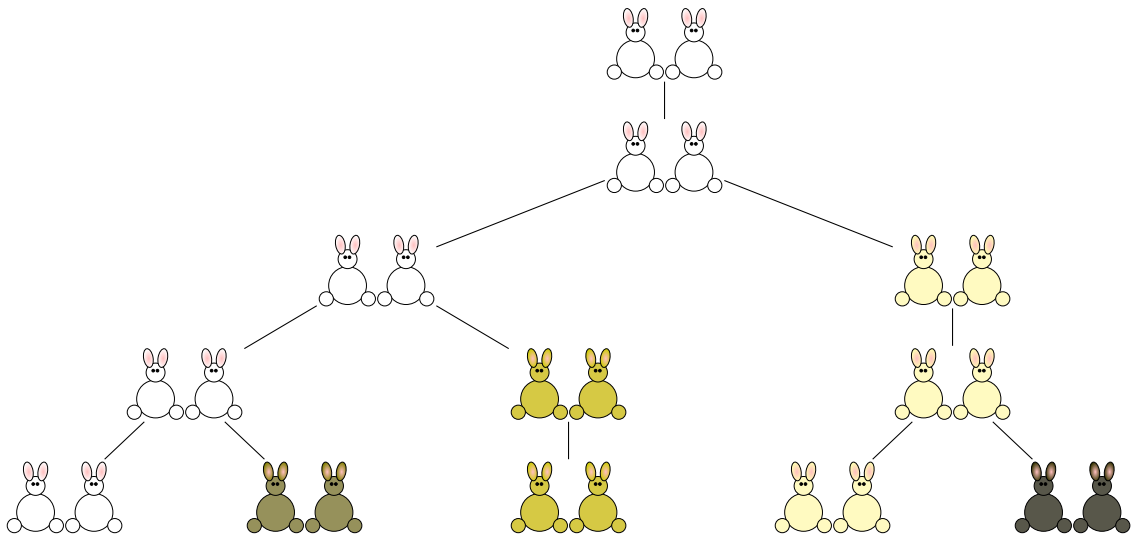
Answer: Pretend silence is a message. Then 1 message can be typed in 0 seconds. One message, $\cdot$, can be typed in 1 second. Two messages take 2 seconds $\cdot\cdot$ or —. A message that can be typed in n seconds is

either an $n-1$ seconds message followed by a $\cdot$ or an $n-2$ second message followed by a —. If you call the number of n-second messages F_n, then

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 1$, $F_1 = 1$.

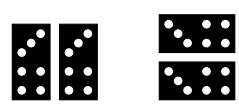
16. (Be able to construct a recurrence equation by examining initial cases.) Let R_n be the number of pairs of rabbits at month n. In the beginning there were no rabbits: $R_0 = 0$. At the beginning of rabbits, God created an original pair of rabbits: $R_1 = 1$. At one month of age the pair mated: $R_2 = 1$. At two months they produced one pair of offspring: $R_3 = 2$. Each pair repeats the same cycle, and rabbits never die.



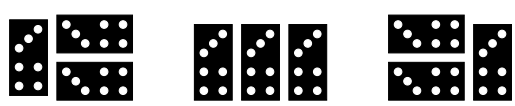
Find a recurrence equation (and initial conditions) that describes the population of rabbits.

17. (Be able to construct a recurrence equation by examining initial cases.) A domino is a rectangular tile 1 unit long and 2 units high

Dominos can be laid out to tile an $n \times 2$ area. For example, there are 2 ways to tile a 2×2 area



And 3 ways to tile a 3×2 area:



Find a recurrence relation that counts the number of $n \times 2$ tilings.

18. (Understand simple interest.) Pretend \$1 is invested on January 2, 2010. Pretend interest accrues at an annual compound rate of 10%. (For example, on January 2, 2011 there will be \$1.10 in the account

and on January 2, 2012 there will be \$1.21 in the account.) How many years will you need to invest to save over \$1,000,000?

Answer: The amount A_n in the account at the start of year n satisfies the recurrence

$$A_n = A_{n-1} + 0.10A_{n-1} = 1.1A_{n-1}$$

This recurrence generates a geometric sequence and in n years, the amount in the account will be

$$A_n = \$1.00(1.1)^n$$

Set this expression equal to \$1,000,000 and solve for n using the logarithm function.

$$\begin{aligned}
\$1.00(1.1)^n &= \$1,000,000 \\
(1.1)^n &= 1,000,000 \\
\log(1.1)^n &= \log 1,000,000 \\
n\log(1.1) &= 6 \\
n &= \frac{6}{\log(1.1)} \\
n &\approx 145 \text{ years}
\end{aligned}$$

19. (Understand compound interest.) Pretend you invested \$1 on January 2 every year. Pretend interest accrues at an annual compound rate of 10%. For example, if \$1 was invested on January 2, 2010, then on January 2, 2011 and January 2, 2012 there will be \$2.10 and \$1.00 + \$2.10 + \$0.21 = \$3.31 in account. How many years will take to save over \$1,000,000?

Answer: The amount A_n in the account at the start of year n satisfies the recurrence

$$A_n = 1.1A_{n-1} + \$1$$

This recurrence generates a sum of a geometric sequence

$$A_0 = 1,\ A_1 = 1.1 + 1,\ A_2 = 1.1^2 + 1.1 + 1,\ A_3 = 1.1^3 + 1.1^2 + 1.1 + 1, \ldots$$

At the beginning of year n, the amount in the account will be

$$\$1.00 + \$1.00(1.1) + \$1.00(1.1)^2 + \cdots + \$1.00(1.1)^n$$

This geometric sum can be evaluated by the function

$$f(n) = \$1.00\left[\frac{(1.1)^{n+1} - 1}{0.1}\right]$$

Set this expression equal to \$1,000,000 and solve for n using the logarithm function.

$$\begin{aligned}
\$10.00(1.1)^{n+1} &= \$1,000,000 \\
(1.1)^{n+1} &= 100,000 \\
\log(1.1)^{n+1} &= \log 100,000 \\
(n+1)\log(1.1) &= 5 \\
n &= \frac{5}{\log(1.1)} - 1 \\
n &\approx 120 \text{ years}
\end{aligned}$$

20. (Understand compound interest.) Pretend you invested \$1 on the second day of every month. Pretend interest accrues at an monthly compound rate of 10%. How many months will you need to wait to save over \$1,000,000?

Answer: At the beginning of month n, the amount in the account will be

$$\$1.00 + \$1.00(1+0.1) + \$1.00(1+0.1)^2 + \cdots + \$1.00(1+0.1)^n$$

This geometric sum can be evaluated by the function

$$f(n) = \$1.00\left[\frac{(1.1)^{n+1}-1}{0.1}\right]$$

Set this expression equal to \$1,000,000 and solve for n using the logarithm function.

$$\begin{aligned}
\$10.00(1.1)^{n+1} &= \$1,000,000 \\
(1.1)^{n+1} &= 100,000 \\
\log(1.1)^{n+1} &= \log 100,000 \\
(n+1)\log(1.1) &= 5 \\
n &= \frac{5}{\log(1.1)} - 1 \\
n &\approx 120\,\text{months}
\end{aligned}$$

That is, in about 10 years!

21. (Know how to construct a recurrence equation by cases.) Consider the sequence generated by adding the alternating integers $\langle 1, -2, 3, -4, 5, \ldots\rangle$. The first few sums of these alternating integers are

$$\langle 1, -1, 2, -2, 3, -3 \ldots\rangle$$

Find a function $f(n)$ that generate these partial sums. That is, $f(0)$ should equal 1, $f(1)$ should equal -1, $f(2)$ should equal 2, $f(3)$ should equal -2, and so on.

Answer: When n is even $f(n)$ is positive, and when n is odd $f(n)$ is negative. For even n

$$\begin{aligned}
f(0) &= f(2\cdot 0) = 0+1 = 1 \\
f(2) &= f(2\cdot 1) = 1+1 = 2 \\
f(4) &= f(2\cdot 2) = 2+1 = 3
\end{aligned}$$

For odd n

$$\begin{aligned}
f(1) &= f(2\cdot 0+1) = -(0+1) = -1 \\
f(3) &= f(2\cdot 1+1) = -(1+1) = -2 \\
f(5) &= f(2\cdot 2+1) = -(2+1) = -3
\end{aligned}$$

$$f(n) = \begin{cases} \frac{n}{2} + 1 & \text{when } n \text{ is even} \\ -\frac{n-1}{2} + 1 & \text{when } n \text{ is odd} \end{cases}$$

If

$$f(n) = \sum_{k=0}^{n} (-1)^k (k+1) = \frac{n}{2} + 1$$

for even n and

$$f(n) = \sum_{k=0}^{n} (-1)^k (k+1) = -\frac{n-1}{2} + 1$$

for odd n, then when n is even

$$\begin{aligned} f(n+1) &= \sum_{k=0}^{n} (-1)^k (k+1) + (-1)^{n+1}(n+2) \\ &= \left(\frac{n}{2} + 1\right) + (-1)^{n+1}(n+2) \\ &= \left(\frac{n}{2} + 1\right) - (n+2) \\ &= -\left(\frac{n}{2} + 1\right) \end{aligned}$$

and when n is odd

$$\begin{aligned} f(n+1) &= \sum_{k=0}^{n} (-1)^k (k+1) + (-1)^{n+1}(n+2) \\ &= -\left(\frac{n-1}{2} + 1\right) + (-1)^{n+1}(n+2) \\ &= -\left(\frac{n}{2} + 1\right) + (n+2) \\ &= \left(\frac{n}{2} + 1\right) \end{aligned}$$

This establishes the proof.

22. (Know how to construct a recurrence equation by cases.) Consider all strings of over the alphabet $\{a, b\}$ listed in order by length then alphabetically, that is,

$$\lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots$$

where λ is the empty string. Name the terms in this sequence s_n so that $s_0 = \lambda$, $s_1 = a = s_0 a$, $s_2 = s_0 b$, $s_3 = s_1 a = aa$, $s_4 = s_1 b = ab$, and so on. Find a recursion equation that models the generation of these alphabetically ordered strings.

Answer: Notice that even subscripted terms can be computed by the recursion

$$\begin{aligned} s_0 &= \lambda \\ s_2 &= s_{2\cdot 1} b = s_0 b = b \\ s_4 &= s_{2\cdot 2} b = s_1 b = ab \\ s_6 &= s_{2\cdot 3} b = s_2 b = bb \end{aligned}$$

In general,

$$s_{2k} = s_{2 \cdot k} b = s_{k-1} b$$

Notice that odd subscripted terms can be computed by the recursion

$$\begin{aligned} s_1 &= s_{2 \cdot 0+1} a = s_0 a \\ s_3 &= s_{2 \cdot 1+1} a = s_1 a \\ s_5 &= s_{2 \cdot 2+1} a = s_2 a \\ s_7 &= s_{2 \cdot 3+1} a = s_3 a \\ &\text{in general} \\ s_{2k+1} &= s_{2 \cdot k+1} b = s_k a \end{aligned}$$

Thus, the alphabet sequence can be enumerated by the cases

$$s_n = \begin{cases} s_{n/2-1} & \text{when } n \text{ is even} \\ s_{(n-1)/2} & \text{when } n \text{ is odd} \end{cases}$$