Relations

Definitions

A relation is a set of ordered pairs \((x, y)\) where \(x\) is “related” to \(y\). Let \(\sim\) denote a relational symbol. Write \(x \sim y\) to express the predicate \(x\) is related to \(y\).

- A relation is reflexive if for all \(x\), \(x \sim x\).
- A relation is symmetric if for all \(x\) and \(y\), \(x \sim y\) implies \(y \sim x\).
- A relation is antisymmetric if for all \(x\) and \(y\), \(x \sim y\) and \(y \sim x\) implies \(x = y\).
- A relation is transitive if for all \(x\), \(y\) and \(z\), \(x \sim y\) and \(y \sim z\) implies \(x \sim z\).

Problems

1. Use Boolean algebra to show that \((x \sim y) \rightarrow (y \sim x)\) is equivalent to \((y \not\sim x) \rightarrow (x \not\sim y)\).

   Answer:

   \[
   \begin{array}{cccccc}
   (x \sim y) & (y \sim x) & (x \sim y) \rightarrow (y \sim x) & (y \not\sim x) & (x \not\sim y) \\
   0 & 0 & 1 & 1 & 1 \\
   0 & 1 & 1 & 0 & 1 \\
   1 & 0 & 0 & 1 & 0 \\
   1 & 1 & 1 & 0 & 0 \\
   \end{array}
   \]

   Note that \((x \sim y) \rightarrow (y \sim x)\) and \((y \not\sim x) \rightarrow (x \not\sim y)\) are two equivalent ways to state \(\sim\) is a symmetric relation.

2. Use Boolean algebra to show that \(((x \sim y) \land (y \sim x)) \rightarrow (x = y)\) is equivalent to \((x \neq y) \rightarrow ((x \not\sim y) \lor (y \not\sim x))\).
Answer:

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<tr>
<th>$x \sim y$</th>
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<th>$x = y$</th>
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Note that $((x \sim y) \land (y \sim x)) \rightarrow (x = y)$ and $(x \not= y) \rightarrow ((x \not\sim y) \lor (y \not\sim x))$
are two equivalent ways to state $\sim$ is an antisymmetric relation.

3. Use Boolean algebra to show that $((x \sim y) \land (y \sim z)) \rightarrow (x \sim z)$
equivalent to $(x \not\sim z) \rightarrow ((x \not= y) \lor (y \not= z))$

Answer:

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Note that $((x \sim y) \land (y \sim z)) \rightarrow (x \sim z)$ and $(x \not\sim z) \rightarrow ((x \not= y) \lor (y \not= z))$
are two equivalent ways to state $\sim$ is a transitive relation.

4. For each relation (on the set $\mathbb{R}$ of real numbers), identify if it is
reflexive, symmetric, antisymmetric, or transitive.

(a) Equality: $x = y$

Answer:
- $x = x$ is True, so equality is reflexive
- If $x = y$, then $y = x$ is True, so equality is symmetric
- If $x = y$ and $y = x$, then $x = y$ is True, so equality is antisymmetric
- If $x = y$ and $y = z$, then $x = z$ is True, so equality is transitive

(b) Not equal: $x \not= y$

Answer:
- $x \not= x$ is False, so not equal is not reflexive
• If $x \neq y$, then $y \neq x$ is True, so not equal is symmetric
• If $x \neq y$ and $y \neq x$, then $x = y$ is False, so equal is not antisymmetric
• If $x \neq y$ and $y \neq z$, then $x \neq z$ is False, so not equality is not transitive

(c) Less than: $x < y$

Answer:
• $x < x$ is False, so less than is not reflexive
• If $x < y$, then $y < x$ is False, so less than is not symmetric
• If $(x < y$ and $y < x)$, then $x = y$ is True, so less than is antisymmetric (The conditional is True because the premise $(x < y$ and $y < x)$ is False)
• If $x < y$ and $y < z$, then $x < z$ is True, so less than is transitive

(d) Greater than or equal: $x \geq y$

Answer:
• $x \geq x$ is True, so greater than or equal is reflexive
• If $x \geq y$, then $y \geq x$ is False, so greater than or equal is not symmetric
• If $(x \geq y$ and $y \geq x)$, then $x = y$ is True, so greater than or equal is antisymmetric
• If $x \geq y$ and $y \geq z$, then $x \geq z$ is True, so greater than or equal is transitive

(e) Equal magnitude: $|x| = |y|

Answer:
• $|x| = |x|$ is True, so equal magnitude is reflexive
• If $|x| = |y|$, then $|y| = |x|$ is True, so equal magnitude is symmetric
• If $(|x| = |y|$ and $|y| = |x|)$, then $x = y$ is False, so equal magnitude is not antisymmetric
• If $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$ is True, so equal magnitude is transitive

(f) Approximately equal (within $\epsilon > 0$): $|x - y| \leq \epsilon$

Answer:
• $|x| = |x|$ is True, so equal magnitude is reflexive
• If $|x| = |y|$, then $|y| = |x|$ is True, so equal magnitude is symmetric
• If $(|x| = |y|$ and $|y| = |x|)$, then $x = y$ is False, so equal magnitude is not antisymmetric
• If $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$ is True, so equal magnitude is transitive

5. For each relation (on the set of geometric figures), identify if it is reflexive, symmetric, antisymmetric, or transitive.

(a) Parallel lines: $L_0 \parallel L_1$
Answer:
• $L_0 \parallel L_0$ is True, so parallel lines is reflexive
• If $L_0 \parallel L_1$, then $L_1 \parallel L_0$ is True, so parallel lines is symmetric
• If $(L_0 \parallel L_1$ and $L_1 \parallel L_0)$, then $L_0 = L_1$ is False, so parallel lines is not antisymmetric
• If $L_0 \parallel L_1$ and $L_1 \parallel L_2$, then $L_0 \parallel L_2$ is True, so parallel lines is transitive

(b) Perpendicular lines: $L_0 \perp L_1$
Answer:
• $L_0 \perp L_0$ is False, so perpendicular lines is not reflexive
• If $L_0 \perp L_1$, then $L_1 \perp L_0$ is True, so perpendicular lines is symmetric
• If $(L_0 \perp L_1$ and $L_1 \perp L_0)$, then $L_0 = L_1$ is False, so perpendicular lines is not antisymmetric
• If $L_0 \perp L_1$ and $L_1 \perp L_2$, then $L_0 \perp L_2$ is False, so perpendicular lines is not transitive

(c) Similar triangles: $T_0 \sim T_1$ (every angle in $T_0$ is equal to a corresponding angle in $T_1$)
Answer:
• $T_0 \sim T_0$ is True, so similar triangles is reflexive
• If $T_0 \sim T_1$, then $T_1 \sim T_0$ is True, so similar triangles is symmetric
• If $(T_0 \sim T_1$ and $T_1 \sim T_0)$, then $T_0 = T_1$ is False, so similar triangles is not antisymmetric
• If $T_0 \sim T_1$ and $T_1 \sim T_2$, then $T_0 \sim T_2$ is True, so similar triangles is transitive

(d) Congruent triangles: $T_0 \cong T_1$ (every angle and side in $T_0$ is equal to a corresponding angle and side in $T_1$)
Answer:
• $T_0 \cong T_0$ is True, so congruent triangles is reflexive
• If $T_0 \cong T_1$, then $T_1 \cong T_0$ is True, so congruent triangles is symmetric
• If $(T_0 \cong T_1$ and $T_1 \cong T_0)$, then $T_0 = T_1$ is False, so congruent triangles is not antisymmetric
• If $T_0 \cong T_1$ and $T_1 \cong T_2$, then $T_0 \cong T_2$ is True, so congruent triangles is transitive