1. (15 pts) Consider the sequence of partial sums of the first $n$ natural numbers

\[ s_n = \sum_{k=0}^{n-1} k = 0 + 1 + 2 + 3 + \cdots + (n - 1) \] with initial condition $s_0 = 0$

(a) What are the first 7 terms in the sequence?
Answer: \[(0, 0, 1, 3, 6, 10, 15)\]

(b) What recurrence equation does the sequence satisfy?
Answer:
\[ s_n = s_{n-1} + (n - 1) \]

(c) What function $f(n)$ computes the value of $s_n$?
Answer:
\[ f(n) = \frac{n(n-1)}{2} \]

2. (10 pts) Use mathematical induction to prove

\[ \sum_{k=0}^{n-1} \binom{k}{m} = \binom{n}{m+1} \]

Answer:
(a) Establish a basis: For $n = 0$ the sum is empty and equal to zero, and the binomial coefficient $\binom{0}{m+1}$ is zero as well. If you don’t like empty proofs: For $n = 1$ the sum is $\binom{0}{m}$ and equal to zero, unless $m = 0$, in which case $\binom{0}{0} = 1$. The binomial coefficient on the right is $\binom{1}{m+1}$ which is equal to zero, unless $m = 0$, in which case $\binom{1}{1} = 1$.

(b) Make a hypothesis: For some $n \geq 0$
\[ \sum_{k=0}^{n-1} \binom{k}{m} = \binom{n}{m+1} \]
(c) Use the hypothesis: Consider the equalities

\[ \sum_{k=0}^{n} \binom{k}{m} = \left( \sum_{k=0}^{n-1} \binom{k}{m} \right) + \binom{n}{m} \quad \text{definition of summations} \]

\[ = \binom{n}{m+1} + \binom{n}{m} \quad \text{by the hypothesis} \]

\[ = \binom{n+1}{m+1} \quad \text{Pascal's identity} \]