1. Use mathematical induction to show that

\[
\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}
\]

Answer:

• Basis: For \( n = 0 \) the sum is \( 0^2 = 0 \) and the function is \( 0(0 + 1)(2 \cdot 0 + 1)/6 = 0 \).

• Hypothesis: Assume for some \( n \geq 0 \) that

\[
\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}
\]

• Prove that

\[
\sum_{k=0}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}
\]

Notice that

\[
\sum_{k=0}^{n+1} k^2 = \sum_{k=0}^{n} k^2 + (n+1)^2
\]

\[
= \frac{n(n+1)(2n+1)}{6} + (n+1)^2
\]

\[
= (n+1) \left[ \frac{n(2n+1)}{6} + (n+1) \right]
\]

\[
= (n+1) \left[ \frac{2n^2 + 7n + 6}{6} \right]
\]

\[
= (n+1) \left[ \frac{(2n+3)(n+2)}{6} \right]
\]

\[
= \frac{(n+1)(n+2)(2n+3)}{6}
\]

2. Use mathematical induction to show that

\[
\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n + 2}{2^n}
\]

Answer:

• Basis: For \( n = 1 \) the sum is \( 1/2 \) and the function is \( 2 - 3/2 = 1/2 \).
• Hypothesis: Assume for some \( n \geq 1 \) that
\[
\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}
\]

• Prove that
\[
\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+3}{2^{n+1}}
\]

Notice that
\[
\sum_{k=1}^{n+1} \frac{k}{2^k} = \sum_{k=1}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}}
\]
\[
= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}}
\]
\[
= 2 - \left(\frac{2n+4}{2^{n+1}} - \frac{n+1}{2^{n+1}}\right)
\]
\[
= 2 - \frac{n+3}{2^{n+1}}
\]

3. Prove that the sum of the odd-indexed Fibonacci numbers up to \( F_{2n+1} \) equals \( F_{2n+2} \). That is,
\[
\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}
\]

Answer: Note the Fibonacci sequences is
\[
\vec{F} = (F_0, F_1, F_2, F_3, \ldots) = (0, 1, 1, 2, \ldots)
\]

• Basis: For \( n = 0 \) the sum is \( F_1 = 1 \) and term on the right is \( F_2 = 1 \).
• Hypothesis: Assume for some \( n \geq 0 \) that
\[
\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}
\]

• Prove that
\[
\sum_{k=0}^{n+1} F_{2k+1} = F_{2n+4}
\]

Notice that
\[
\sum_{k=0}^{n+1} F_{2k+1} = \sum_{k=0}^{n} F_{2k+1} + F_{2n+3}
\]
\[
= F_{2n+2} + F_{2n+3}
\]
\[
= F_{2n+4}
\]

4. Prove that
\[
\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_{n+1}
\]

Answer:
• Basis: For $n = 0$ the sum is $\binom{0}{0} = 1$ and the term on the right is $F_1 = 1$. Because the Fibonacci recurrence is second-order, we need a second case here, as you will see in a moment. For $n = 1$ the sum is $\binom{1}{0} + \binom{0}{1} = 1 + 0 = 1$ and the term on the right is $F_2 = 1$.

• Hypothesis: Assume for some $n \geq 1$ that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_{n+1}$$

and

$$\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots = F_n$$

• Prove that

$$\binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots = F_{n+2}$$

Notice that, by Pascal’s identity

$$\binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots = \binom{n+1}{0} + \left[ \binom{n}{1} + \left( \binom{n-1}{1} + \binom{n-2}{2} + \cdots \right) \right] + \cdots$$

$$= \binom{n}{0} + \left[ \binom{n-1}{0} + \binom{n-1}{1} + \cdots \right]$$

$$= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

$$= F_{n+1} + F_n$$

$$= F_{n+2}$$

5. Prove that $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ solves the recurrence equation

$$H_{n+1} = H_n + \frac{1}{n+1}, \quad H_0 = 0$$

Answer: If

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

then

$$H_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}$$

or

$$H_{n+1} = H_n + \frac{1}{n+1}, \quad H_0 = 0$$

Also note the sum for $H_n$ is undefined when $n = 0$, so it must be empty and equal to 0.

Total Points: 0