Name: **Class ID:**

CSE 1400 Applied Discrete Mathematics
Fall 2013 Midterm

DMonday, October 7, 2013

1. Use mathematical induction to show that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Answer:

- Basis: For $n = 0$ the sum is $0^2 = 0$ and the function is $0(0+1)(2 \cdot 0 + 1)/6 = 0$.
- Hypothesis: Assume for some $n \geq 0$ that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

- Prove that

$$\sum_{k=0}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Notice that

$$\begin{aligned}
\sum_{k=0}^{n+1} k^2 &= \sum_{k=0}^{n} k^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= (n+1)\left[\frac{n(2n+1)}{6} + (n+1)\right] \\
&= (n+1)\left[\frac{2n^2+7n+6}{6}\right] \\
&= (n+1)\left[\frac{(2n+3)(n+2)}{6}\right] \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

2. Use mathematical induction to show that

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

Answer:

- Basis: For $n = 1$ the sum is $1/2$ and the function is $2 - 3/2 = 1/2$.

- Hypothesis: Assume for some $n \geq 1$ that

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

- Prove that

$$\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+3}{2^{n+1}}$$

Notice that

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{k}{2^k} &= \sum_{k=1}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}} \\
&= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \\
&= 2 - \left(\frac{2n+4}{2^{n+1}} - \frac{n+1}{2^{n+1}} \right) \\
&= 2 - \frac{n+3}{2^{n+1}}
\end{aligned}$$

3. Prove that the sum of the odd-indexed Fibonacci numbers up to F_{2n+1} equals F_{2n+2}. That is,

$$\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}$$

Answer: Note the Fibonacci sequences is

$$\vec{F} = \langle F_0,\, F_1,\, F_2,\, F_3, \ldots \rangle = \langle 0,\, 1,\, 1,\, 2, \ldots \rangle$$

- Basis: For $n = 0$ the sum is $F_1 = 1$ and term on the right is $F_2 = 1$.
- Hypothesis: Assume for some $n \geq 0$ that

$$\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}$$

- Prove that

$$\sum_{k=0}^{n+1} F_{2k+1} = F_{2n+4}$$

Notice that

$$\begin{aligned}
\sum_{k=0}^{n+1} F_{2k+1} &= \sum_{k=0}^{n} F_{2k+1} + F_{2n+3} \\
&= F_{2n+2} + F_{2n+3} \\
&= F_{2n+4}
\end{aligned}$$

4. Prove that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_{n+1}$$

Answer:

- Basis: For $n = 0$ the sum is $\binom{0}{0} = 1$ and the term on the right is $F_1 = 1$. Because the Fibonacci recurrence is second-order, we need a second case here, as you will see in a moment. For $n = 1$ the sum is $\binom{1}{0} + \binom{0}{1} = 1 + 0 = 1$ and the term on the right is $F_2 = 1$.
- Hypothesis: Assume for some $n \geq 1$ that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_{n+1}$$

and

$$\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots = F_n$$

- Prove that

$$\binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots = F_{n+2}$$

Notice that, by Pascal's identity

$$\begin{aligned}
\binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots &= \binom{n+1}{0} + \left[\binom{n}{1}\right] + \left[\binom{n-1}{2}\right] + \cdots \\
&= \binom{n}{0} + \left[\binom{n-1}{0} + \binom{n-1}{1}\right] + \left[\binom{n-2}{1} + \binom{n-2}{2}\right] + \cdots \\
&= \left[\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots\right] \\
&= \qquad + \left[\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots\right] \\
&= F_{n+1} + F_n \\
&= F_{n+2}
\end{aligned}$$

5. Prove that $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ solves the recurrence equation

$$H_{n+1} = H_n + \frac{1}{n+1}, \quad H_0 = 0$$

Answer: If

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

then

$$H_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}$$

or

$$H_{n+1} = H_n + \frac{1}{n+1}, \quad H_0 = 0$$

Also note the sum for H_n is undefined when $n = 0$, so it must be empty and equal to 0.

Total Points: 0