Name: **Class ID:**

CSE 1400 Applied Discrete Mathematics
Fall 2013 Midterm

Score

1. (10 pts) Let P, Q, and R be Boolean variables. If P = `True`, Q = `False`, and R = `False`, what are the values of the following expressions?

   (a) $\neg(Q \lor R) = \neg($`False` $\lor$ `False`$) =$ `True`

   (b) $\neg(P \land Q) \lor R = \neg($`True` $\land$ FALSE$) \lor$ `False` $=$ `True` $\lor$ `False` $=$ `True`

Score

2. (5 pts) In how many ways can truth values be assigned to n Boolean variables?

   Answer: There are 2^n ways.

Score

3. (5 pts) How many different n-variable Boolean functions are there? There are 2^{2^n} different n-variable Boolean functions.

Score

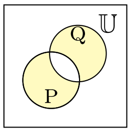
4. (10 pts) Construct a truth table for the Boolean expression (function)

$$((\neg P \to Q) \land (\neg P \to \neg Q)) \to P \tag{1}$$

Input		Output				
P	Q	$((\neg P \to Q)$	$\land$	$(\neg P \to \neg Q))$	$\to$	P
0	0	0	0	1	1	0
0	1	1	0	0	1	0
1	0	1	1	1	1	1
1	1	1	1	1	1	1

Score

5. (5 pts) What Boolean expression does the shaded region represent?



Answer: The shaded region represents $(\text{P} \wedge \neg\text{Q}) \vee (\neg\text{P} \wedge \text{Q})$.

Score

6. (15 pts) Let

$$\begin{aligned}\mathbb{D} &= \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\} \text{ be the universe of digits}\\ \mathbb{E} &= \{0,\ 2,\ 4,\ 6,\ 8\} \text{ be the even digits}\\ \mathbb{O} &= \{1,\ 3,\ 5,\ 7,\ 9\} \text{ be the odd digits}\\ \mathbb{P} &= \{2,\ 3,\ 5,\ 7\} \text{ be the prime digits.}\end{aligned}$$

(a) Is $\mathbb{O} \cup (\mathbb{E} \cap \mathbb{P}) = (\mathbb{O} \cup \mathbb{E}) \cap \mathbb{P}$ `True` or `False`? Explain your answer.
Answer:

$$\begin{aligned}\mathbb{O} \cup (\mathbb{E} \cap \mathbb{P}) &= \{1,\ 3,\ 5,\ 7,\ 9\} \cup \{2\}\\ &= \{1,\ 2,\ 3,\ 5,\ 7,\ 9\}\\ (\mathbb{O} \cup \mathbb{E}) \cap \mathbb{P} &= \{2, 3,\ 5,\ 7\}\end{aligned}$$

So no, the order of set operations matters. The order of precedences is (1) Set complement; (2) Intersect; (3) Union. Use parenthesis to avoid remembering and to clarify an expression.

(b) Verify De Morgan's law: $\neg(\mathbb{O} \cap \mathbb{P}) = \neg\mathbb{O} \cup \neg\mathbb{P}$
Answer:

$$\begin{aligned}\mathbb{O} \cap \mathbb{P} &= \{3,\ 5,\ 7\}\\ \neg(\mathbb{O} \cap \mathbb{P}) &= \{0,\ 1,\ 2,\ 4,\ 6,\ 8,\ 9\}\\ \neg\mathbb{O} &= \mathbb{E}\\ \neg\mathbb{P} &= \{0,\ 1,\ 4,\ 6,\ 8,\ 9\}\\ \neg\mathbb{O} \cup \neg\mathbb{P} &= \{0,\ 1,\ 2,\ 4,\ 6,\ 8,\ 9\}\end{aligned}$$

(c) What is (describe) $2^{\mathbb{D}}$?
Answer: $2^{\mathbb{D}}$ is the set of all subsets of $\mathbb{D}$: The power set of $\mathbb{D}$.

Score

7. (10 pts) I once gave a 20 question `True`/`False` exam.

(a) In how many ways can you answer the questions (pretend you answer each question `True` or `False`)?
Answer: There are 2^{20} ways: There are two answers (`True` or `False`) that can be chosen for each question.

(b) If you decide to answer one-half of the questions `True` and one-half `False`, in how many ways can you answer the questions?
Answer: There are

$$\binom{20}{10} = \frac{20!}{10!10!} \quad \text{ways.}$$

Choose 10 of the 20 questions to answer `True`. The other 10 will be answered `False`. There are $\binom{20}{10}$ ways to choose 10 questions from 20.

Score

8. (10 pts) What is Pascal's identity and what are its boundary conditions?

Answer: Pascal's identity is

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and the boundary conditions are

$$\binom{n}{0} = \binom{n}{n} = 1$$

Score

9. (5 pts) Let $\mathbb{X} = \{a, b, c, d, e\}$. What is the notation for the Stirling number of the second kind, which counts partitions of $\mathbb{X}$ into 3 subsets? (Extra credit (5 pts)) What is the Stirling's second recurrence equation for counting partitions?

Answer: The Stirling number of the second kind $\left\{ {5 \atop 3} \right\}$. Although not required for full marks, note that

$$\begin{aligned}
\left\{ {5 \atop 3} \right\} &= 3 \cdot \left\{ {4 \atop 3} \right\} + \left\{ {4 \atop 2} \right\} \\
&= 3 \cdot \left(3 \cdot \left\{ {3 \atop 3} \right\} + \left\{ {3 \atop 2} \right\} \right) + \left(2 \left\{ {3 \atop 2} \right\} + \left\{ {3 \atop 1} \right\} \right) \\
&= 3 \cdot (3 \cdot 1 + 3) + (2 \cdot 3 + 1) \\
&= 18 + 7 \\
&= 25
\end{aligned}$$

The recurrence is

$$\left\{ {n \atop k} \right\} = k \cdot \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$$

Score

10. (10 pts) Consider the sequence $\vec{M} = \langle 0, 1, 3, 7, 15, 31, 63, \ldots \rangle$.

(a) What is the function $m(n)$ that computes the terms m_n for $n \in \mathbb{N}$?
Answer: The terms can be computed by the function $m(n) = 2^n - 1$.

(b) What is the recurrence equation that terms in the sequence satisfy?
Answer: The recurrence equation is $m_n = 2m_{n-1} + 1$.

Score

11. (5 pts) Write the first 6 terms in the sequence defined by the recurrence equation and initial condition.

$$f_n = f_{n-1} - f_{n-2}, \qquad f_0 = 0,\ f_1 = 1$$

Answer: $f_0 = 0,\ f_1 = 1,\ f_2 = f_1 - f_0 = 1,\ f_3 = f_2 - f_1 = 0,\ f_4 = f_3 - f_2 = -1,\ f_5 = f_4 - f_3 = -1.$

Score

12. (5 pts) Consider the summation

$$S_n = \sum_{k=0}^{n-1} (4k+3)$$

Show that the function $S(n) = n(2n+1)$ satisfies the recurrence equation $S_{n+1} = S_n + (4n+3)$.

Answer: If $S(n) = n(2n+1)$, then $S(n+1) = (n+1)(2n+3)$ and

$$\begin{aligned} S(n) + (4n+3) &= (2n^2 + 2n) + (4n+3) \\ &= 2n^2 + 6n + 3 \\ &= (n+1)(2n+3) \end{aligned}$$

Score

13. (5 pts) Use mathematical induction to prove the following summation formula.

$$\sum_{k=0}^{n-1} k(k-1) = \frac{n(n-1)(n-2)}{3}$$

Answer:

- Basis: For $n = 0$, the sum on the left is empty and equal to 0. Likewise, the function on the right is clearly equal to 0 when $n = 0$.
- Hypothesis: Pretend

$$\sum_{k=0}^{n-1} k(k-1) = \frac{n(n-1)(n-2)}{3}$$

  is `True` for some $n \geq 0$.
- To Prove: Adding another term to the sum preserves the form of the function. Consider the sequence of equations.

$$\begin{aligned} \sum_{k=0}^{n} k(k-1) &= \sum_{k=0}^{n-1} k(k-1) + n(n-1) \\ &= \frac{n(n-1)(n-2)}{3} + n(n-1) \\ &= n(n-1)\left[\frac{n-2}{3} + 1\right] \\ &= \frac{(n+1)(n)(n-1)}{3} \end{aligned}$$