1. (30 pts) Consider the sets

\[ Z = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots \} \quad N = \{0, 1, 2, 3, 4, 5, \ldots \} \]

of integers and natural numbers. Let \( a, b \in Z \) and let \( n \in N \). If \( n \) divides \( a - b \), write

\[ a \equiv b \pmod{n} \]

and say “\( a \) is congruent to \( b \) modulo \( n \).”

Show that congruence modulo \( n \) is an equivalence relation.

**Answer:** Congruence is

- Reflexive: \((\forall a \in Z)(a \equiv a \pmod{n})\) because \( n \) divides \( a - a = 0 \).
- Symmetric: \((\forall a, b \in Z)((a \equiv b \pmod{n}) \rightarrow (b \equiv a \pmod{n}))\) because if \( n \) divides \( a - b \), then \( n \) divides \( b - a \). That is, if \( a - b = n \cdot q \) for some quotient \( q \in Z \), then \( b - a = n(-q) \).
- Transitive: \((\forall a, b, c \in Z)((a \equiv b \pmod{n}) \land (b \equiv c \pmod{n}) \rightarrow (a \equiv c \pmod{n}))\) because if \( n \) divides \( a - b \) and \( n \) divides \( b - c \), then \( n \) divides \( a - c \). That is, if \( a - b = n \cdot q \) for some quotient \( q \in Z \) and \( b - c = n \cdot q' \), then \( a - c = (a - b) + (b - c) = n(q + q') \), and \( n \) divides \( a - c \).

2. (20 pts) Use Horner’s rule to evaluate the polynomial \( p(x) = 4x^3 - 5x + 7 \) at \( x = 2 \).

**Answer:**

<table>
<thead>
<tr>
<th>Horner’s Rule</th>
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<tbody>
<tr>
<td>4</td>
</tr>
<tr>
<td>8</td>
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<tr>
<td>4</td>
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3. (20 pts) Show that $T(n) = n \log_2 n$ solves the recurrence equation

$$T_n = 2T_{n/2} + n, \quad T_1 = 0$$

For $n = 1, 2, 4, 8, 16, \ldots$.

Answer: First, $T(1) = 1 \log 1 = 0 = T_1$. Next, if

$$T_{2^n} = \frac{n}{2} \log \left(\frac{n}{2}\right)$$

for some $n = 2^p \geq 1$, then

$$2T_{2^n} + n = n \log \left(\frac{n}{2}\right) + n$$

$$= n(\log n - \log 2) + n$$

$$= n(\log n - 1) + n$$

$$= n \log n$$

$$= T_n$$

4. (20 pts) Consider the set $\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) Write the permutation $(9, 7, 5, 3, 1, 0, 2, 4, 6, 8)$ of elements in $\mathbb{D}$ using cyclic notation.

Answer: Write the elements in their natural order above their permuted order:

<table>
<thead>
<tr>
<th>0 1 2 3 4 5 6 7 8 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>9 7 5 3 1 0 2 4 6 8</td>
</tr>
</tbody>
</table>

Then read off the movement of elements

$[0, 5, 2, 6, 8, 9][1, 4, 7][3]$

(b) How many cycles are there in the permutation?

Answer: There are 3 cycles.

5. (10 pts) How many bits are needed to write 573 in binary?

Answer: The number of bits needed to represent $n$ is given by the formula

$$\lceil \log_2 n \rceil + 1$$

In this case, since $512 = 2^9 < 573 < 2^{10} = 1024$, the floor of the log (base 2) of 573 is 9. Therefore 10 bits are required to write 573 in binary.