1. (20 pts) What is the largest natural number, call it $m$, that can be written using $n$ bits?

Answer: The largest $n$-bit number is

$$m = 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$$

Notice $m$ is a Mersenne number.

2. (10 pts) Let $x$ be a positive natural number. How many bits are needed to write $x$ in binary notation?

Answer: If $x$ is between $2^n - 1$ and $2^{n+1} - 1$, that is $2^n - 1 \leq x \leq 2^n - 1$, then $x$ can be written using $n$ bits. For instance, If $8 = 2^3 \leq x \leq 2^4 - 1 = 15$, then $m$ can be written in 4-bits: $(1000)_2 = 8$ through $(1111)_2 = 15$.

You can solve the inequality $2^n - 1 \leq x \leq 2^n - 1$ for $n$ as follows

$$2^n - 1 \leq x \leq 2^n - 1$$

$$n - 1 \leq \lg x < n$$

Therefore, $n - 1 = \lfloor \lg x \rfloor$ and $n = \lfloor \lg x \rfloor + 1$ bits are needed to represent $x$.

3. (20 pts) Let $(11001000)_2$ be an (8-bit, unsigned) natural number, written in binary notation. Use Horner’s rule to convert $(11001000)_2$ into decimal notation.

Answer:

<table>
<thead>
<tr>
<th>Horner’s Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Therefore, $(11001000)_2 = (200)_{10}$
4. Let \((11001000)_{2c}\) be an (8-bit, signed) integer, written in two’s complement notation.

(a) (5 pts) Is \((11001000)_{2c}\) positive or negative?

Answer: The number is negative because the leading (left-most) bit is 1.

(b) (10 pts) What is the negative of \((11001000)_{2c}\)? That is, negate \((11001000)_{2c}\).

Answer: Using the rule: Copy all bits from right-to-left up to and including the first 1, then flip the remaining bits, the negative of \((11001100)_{2c}\) is \((00110100)_{2c}\).

(c) (10 pts) What is the decimal representation of \((11001000)_{2c}\)?

Answer: The easy answer is to use the result of the previous problem: An 8-bit two’s complement number and its negative sum to 256. Therefore, since \((11001100)_{2c}\) is a negative number, its value is \(204 - 256 = -52\).

5. (15 pts) Convert the decimal number \(-56\) into its two’s complement representation.

Answer: Follow the process:

(a) Use repeated division by 2 on the absolute value \(|\ -37\ | = 37\) to compute remainder bits from least to most significant.

\[
\begin{array}{cccccc}
\text{REPEATED REMAINDERING MOD 2} \\
\text{QUOTIENTS} & 37 & 18 & 9 & 4 & 2 & 1 \\
\text{REMAINERS} & 1 & 0 & 1 & 0 & 0 & 1
\end{array}
\]

Therefore, unsigned 37 is \((100101)_{2}\).

(b) Append a leading 0 to the result of step (5a). That is, signed 37 is

\[+37 = (0100101)_{2c}\]

(c) Negate the result of step (5b)

\[-37 = (1011011)_{2c}\]

6. (10 pts) Convert the fixed point (unsigned) binary number \((11.0111)_{2}\) into its decimal (fractional) representation.

Answer:

\[
\begin{array}{cccccc}
\text{HORNER’S RULE} \\
1 & 1 & 0 & 1 & 1 & 1 \\
2 & 6 & 12 & 26 & 54 \\
1 & 3 & 6 & 13 & 27 & 55
\end{array}
\]

Therefore

\[(11.0111)_{2} = \frac{55}{16}\]

Total Points: 100

Friday, November 15, 2013