1. Consider the set of integers 
\[ \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots\} \].

and the set of natural numbers 
\[ \mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\} \].

Let \( a, b \in \mathbb{Z} \) and let \( n \in \mathbb{N} \). If \( n \) divides \( a - b \), write 
\[ a \equiv b \pmod{n} \]
and say \( a \) is congruent to \( b \) modula \( n \).

• Show that congruence modulo \( n \) is an equivalence relation.

Answer: Congruence is

- Reflexive: \((\forall a \in \mathbb{Z})(a \equiv a \pmod{n})\) because \( n \) divides \( a - a = 0 \).
- Symmetric: \((\forall a, b \in \mathbb{Z})(a \equiv b \pmod{n}) \rightarrow (b \equiv a \pmod{n})\) because if \( n \) divides \( a - b \), then \( n \) divides \( b - a \). That is, if \( a - b = n \cdot q \) for some quotient \( q \in \mathbb{Z} \), then \( b - a = n(-q) \).
- Transitive: \((\forall a, b, c \in \mathbb{Z})(a \equiv b \pmod{n}) \land (b \equiv c \pmod{n}) \rightarrow (a \equiv c \pmod{n})\) because if \( n \) divides \( a - b \) and \( n \) divides \( b - c \), then \( n \) divides \( a - c \). That is, if \( a - b = n \cdot q \) for some quotient \( q \in \mathbb{Z} \) and \( b - c = n \cdot q' \), then \( a - c = (a - b) + (b - c) = n(q + q') \), and \( n \) divides \( a - c \).

• What does \( a \equiv b \pmod{1} \) mean?

Answer: If \( a \equiv b \pmod{1} \) then \( a - b \) is divisible by \( 1 \). Since \( 1 \) divides every number, \( a \equiv b \pmod{1} \) means every integer \( a \) is related to every integer \( b \).

• What does \( a \equiv b \pmod{0} \) mean?

Answer: If \( a \equiv b \pmod{0} \) then \( a - b \) is divisible by \( 0 \). The only number \( 0 \) divides is \( 0 \), so \( a \equiv b \pmod{0} \) means \( a = b \).