Name:

CSE 1400 Applied Discrete Mathematics

Fall 2014 Midterm Exam

Score

1. (10 pts) Convert the two's complement number $(1101\ 1000)_{2c}$ to its decimal representation.

   Answer: You could: Negate, convert, and negate.

$$(1101\ 1000)_{2c} \overset{\text{negate}}{\mapsto} (0010\ 1000)_{2c}$$

$$(0010\ 1000)_{2c} \overset{\text{convert}}{\mapsto} (32+8) = 40$$

$$40 \overset{\text{negate}}{\mapsto} -40$$

   Therefore, $(0010\ 1000)_{2c} = -40$.

   It's good to know other algorithms for conversion from two's complement to decimal.

Score

2. (10 pts) Convert -123 to its two's complement representation.

   Answer: You could: Negate, convert to unsigned, pad with sign, and negate.

$$-123 \overset{\text{negate}}{\mapsto} 123$$

   Convert

Repeated Remaindering mod 2

Quotients	123	61	30	15	7	3	1
Remainders	1	1	0	1	1	1	1

   That is, <u>unsigned</u> 123 is binary

$$123 = (111\ 1011)_2$$

   Pad with sign,

$$+123 = (0111\ 1011)_{2c}$$

   Now negate: <u>Signed</u> -123 is two's complement $(1000\ 0101)_{2c}$.

$$-123 = (1000\ 0101)_{2c}$$

   It's good to know other algorithms for conversion from decimal to two's complement.

Score

3. (10 pts) Using the pidgin notation for floating point numbers described in class, what is the decimal value of the normalized floating point number $r = (1\ 011\ 0011)_{fp}$?

   Answer: The normalized floating point number is written as

$$r = (-1)^s\,(1.f)_2 \times 2^e$$

   Where the sign bit $s = 1$ is the leading, leftmost bit.

   The exponent is

$$e = ((011)_2)_{\text{bias}=3} = (3)_{\text{bias}=3} = 3 - 3 = 0$$

   And, the normalized fraction is

$$(1.f)_2 = (1.0011)_2 = \frac{19}{16}$$

   Therefore,

$$r = (1\ 011\ 0011)_{fp} = -\frac{19}{16}$$

Score

4. (12 pts) There was an old woman who lived in a shoe. She had so many children, ..., 23 in fact.

A child is born on some day:

$$\mathbb{DAYS} = \{\text{Monday, Tuesday, } \ldots, \text{ Saturday, Sunday}\}.$$

Let $\textsc{births}(d)$ be the number of births on day $d \in \mathbb{DAYS}$.

(a) What is the quotient q and remainder r when the number of children is divided by the number of days?

Answer: You know from arithmetic

$$23 = 7 \cdot 3 + 2$$

The quotient is $q = 3$ and the remainder is $r = 2$.

(b) Write, in English, the meaning of the predicate statement

$$\textsc{p}(n) = (\exists\ \mathrm{d} \in \mathbb{DAYS})(\textsc{births}(d) \leq n).$$

Answer: There is a day with n or fewer births.

(c) What is the smallest value of n for which the statement

$$\textsc{p}(n) = (\exists\ \mathrm{d} \in \mathbb{DAYS})(\textsc{births}(d) \leq n)) \quad \text{is certain to be } \texttt{True}.$$

Answer: Interpret this as a pigeonhole problem where pigeons are children and holes are the days on which they were born.

There must be a day with $n = 3$ or fewer births. Notice that n is the quotient q when 23 is divided by 7 and

$$n = q = \left\lfloor \frac{23}{7} \right\rfloor$$

If there were not 3 or fewer births or some day, there would be 4 or more births every day, leading to more than 23 children.

There could be days with 2 or fewer births, but you cannot be certain they exists. This is because 23 can be written as

$$23 = 3 + 3 + 3 + 3 + 3 + 4 + 4$$

(d) What is the largest value of m for which the statement

$$\textsc{p}(m) = (\exists\ \mathrm{d} \in \mathbb{DAYS})(\textsc{births}(d) \geq m)) \quad \text{is certain to be } \texttt{True}.$$

Answer: Use the pigeonhole principle again.

There must be a day with $m = 4$ or more births. Notice that m is ceiling of 23 is divided by 7

$$m = \left\lceil \frac{23}{7} \right\rceil$$

If there were not 4 or more births on some day, there there would be 3 or fewer births every day, leading to less than 23 children.

There could be days with more than 4 births, but you cannot be certain they exist.

Score

5. (10 pts) Use Horner's rule to evaluate the polynomial $p(x) = x^4 + 2x^2 - 5x - 11$ at $x = 3$.

Answer:

Horner's Rule @ $x = 3$				
1	0	2	-5	-11
	3	9	33	84
1	3	11	28	$\boxed{73}$

Therefore $p(3) = 73$.

Score

6. (10 pts) The time complexity of Strassen's matrix multiplication algorithm can be modeled by the function

$$T(n) = n^{\lg 7}$$

What is a simple expression for the value of $T(256) = 256^{\lg 7}$?

Answer: Use the fact that $256 = 2^8$ to write

$$\begin{aligned} T(256) &= 256^{\lg 7} \\ &= \left(2^8\right)^{\lg 7} \\ &= \left(2^{\lg 7}\right)^8 \\ &= 7^8 \\ &= 5,764,801 \text{ you are not expected to compute this} \end{aligned}$$

The more general result

$$n^{\log_b m} = m^{\log_b n} \quad \text{is } \texttt{True}.$$

Simply take the log base b of both sides to see the equivalence.

Score

7. (10 pts) Construct a truth table for the three Boolean expressions

$$(\mathrm{P} \to \neg\mathrm{Q}) \qquad (\neg\mathrm{P} \vee \neg\mathrm{Q}) \qquad \neg(\mathrm{P} \wedge \mathrm{Q})$$

What can you conclude about these expressions?

Answer: The expressions are equivalent: They equal each other on all truth assignments.

Input		Output				
P	Q	$(\neg\mathrm{P} \vee \neg\mathrm{Q})$	$\equiv$	$(\mathrm{P} \to \neg\mathrm{Q})$	$\equiv$	$\neg(\mathrm{P} \wedge \mathrm{Q})$
0	0	1	1	1	1	1
0	1	1	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	1	0

Score

8. (10 pts) Pretend you went on a bird census for your biology class. You classified the birds you saw as:

- Small, recorded in set $\mathbb{S}$.
- Gray, recorded in set $\mathbb{G}$.
- Water fowl, recorded in set $\mathbb{W}$.
- Other: not Water fowl, not Small, not Gray.

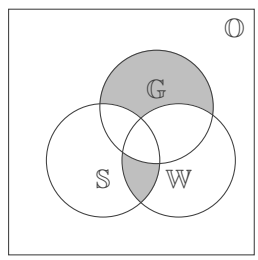
(a) Use set notation to write and expression for the set of bird sightings that were:

$\mathbb{A} =$ (Gray, but neither Small nor Water fowl) or (Small and Water fowl, but not Gray).

Answer: The set is

$$\mathbb{A} = (\mathbb{G} \cap \neg\mathbb{S} \cap \neg\mathbb{W}) \cup (\mathbb{S} \cap \mathbb{W} \cap \neg\mathbb{G})$$

(b) Shade the set $\mathbb{A}$ described above.
Answer:



Score

9. (18 pts) Here are some miscellaneous questions requiring short answers.

(a) How many truth assignments can be made on n Boolean variables?
Answer: There are 2^n truth assignments.

(b) How many Boolean functions can be defined on n Boolean variables?
Answer: There are 2^{2^n} Boolean functions.

(c) How many bits does it take to write the natural number $n > 1$?
Answer: It requires $\lfloor \lg n \rfloor + 1$ bits.

(d) Is $\emptyset \in \emptyset$? Explain your answer.
Answer: No, the empty set is not a member of (element in) the empty set. The empty set contains nothing.

(e) Is $\emptyset \subseteq \emptyset$? Explain your answer.
Answer: Yes, the empty set is a subset of every set, including itself. The conditional statement

$$(x \in \emptyset) \to (x \in \emptyset) \quad \text{is logically } \texttt{True} \text{ for any } x$$

That is because the assumption of the condition: $(x \in \emptyset)$ is `False`.

(f) What integer is represented by the biased number $(73)_{\text{bias}=64}$?
Answer:

$$(73)_{\text{bias}=64} = 73 - 64 = 9$$

(g) What is the negative of the four's complement number $(312)_{4c}$?
Answer: A number and its negative add to 0. In this case, arithmetic is base 4.

1	1	1	0	carry-in
	3	1	2	number
	0	2	2	its negative
	0	0	0	sum
	1	1	1	carry-out

Therefore $-(312)_{4c} = (022)_{4c} = 2 \cdot 4 + 2 = 10$.

(h) What is the value of $\lfloor \lg 35 \rfloor$?
Answer: Since $2^5 < 35 < 2^6$, it follows that $5 < \lg 35 < 6$ and $\lfloor \lg 35 \rfloor = 5$.

(i) What is the name of the sequence $\langle 0, 0, 1, 3, 6, 10, 15, \ldots \rangle$ and what is its sequence of differences?
Answer: This is the <u>triangular</u> sequence and its difference is the Gauss sequence $\langle 0, 1, 2, 3, 4, 5, 6, \ldots \rangle$.

Total Points: 100 Monday, October 6, 2014