Name:

CSE 1400 Applied Discrete Mathematics

Fall 2014 Quiz 1.1

Score

1. (15 pts) Pretend you want to assign identification numbers (IDs) to a group of people. If your IDs are n numerals long, how many people can you name using the sets below?

   (a) $\mathbb{B}$

   Answer: There are 2^n bit strings.

   (b) $\mathbb{D}$

   Answer: There are 10^n decimal strings.

   (c) $\mathbb{H}$

   Answer: There are 16^n hexadecimal strings.

   The inverse problem is: Given the number of people, let's call it m, how long will IDs be?

   To answer the question: Notice that any positive integer $m > 1$ can be bounded by powers of a base. For instance, in binary

   $$1 < m \leq 2, \quad 2 < m \leq 4, \quad 2 < m \leq 8, \ldots, 2^{n-1} < m \leq 2^n$$

   In these cases, it will take IDs of length 1, 2, 3, $\ldots n$.

   Use these bounds and take the base 2 logarithm of the inequalities to get

   $$(n-1) < \lg m \leq n$$

   Round $\lg m$ up to its ceiling n to compute the length n.

   $$\lceil \lg m \rceil = n$$

   For different bases use different logarithm bases.

   It is good to notice the similarities and differences between this problem and problem 6.

Score

2. (10 pts) You are to design a Boolean function that has 3 input variables: P, Q, and R.

   (a) How many different input combinations are there?

   Answer: There are $2^3 = 8$ different inputs: From 000 to 111.

   This number, 8, is the size (cardinality) of the input space, often called the domain of the function and labeled $\mathbb{X}$

(b) How many different functions can be designed?

Answer: There are $2^{2^3} = 2^8 = 256$ different functions. (You should be able to compute $2^8 = 256$. It is the size of the address space of a byte.)

For each input combination there are two possible outputs: `False` or `True`. Therefore there are 2^8 different functions.

The number 2 is the size of the output space $\{0, 1\} = \{$`False`, `True`$\}$, often called the <u>co-domain</u> or <u>range</u> of the function and labeled $\mathbb{Y}$.

The number of functions can be computed by the expression

$$2^8 = |\mathbb{Y}|^{|\mathbb{X}|}$$

where $|\mathbb{X}|$ is the size (cardinality) of the domain and $|\mathbb{Y}|$ is the size (cardinality) of the co-domain.

Score

3. (10 pts) Consider the Boolean expression

$$\neg(\mathrm{P} \wedge \neg\mathrm{Q}) \equiv \neg\mathrm{P} \vee \mathrm{Q}$$

(a) Fill in the truth table for it.

Input		Computations			
P	Q	$\neg$	$(\mathrm{P} \wedge \neg\mathrm{Q})$	$\equiv$	$\neg\mathrm{P} \vee \mathrm{Q}$
0	0	1	0	1	1
0	1	1	0	1	1
1	0	0	1	1	0
1	1	1	0	1	1

(b) What law is being used to establish the equivalence?

Answer: This is an application of one of De Morgan's laws. They are usually stated like

$$\neg(\mathrm{P} \vee \mathrm{Q}) \equiv \neg\mathrm{P} \wedge \neg\mathrm{Q}$$

or

$$\neg(\mathrm{P} \wedge \mathrm{Q}) \equiv \neg\mathrm{P} \vee \neg\mathrm{Q}$$

The question is a minor modification.

Score

4. (15 pts) Consider the Boolean expression.

$$(\mathrm{P} \rightarrow \mathrm{Q}) \rightarrow (\mathrm{P} \rightarrow (\mathrm{P} \wedge \mathrm{Q}))$$

(a) Construct a truth table for it.

Answer:

Input		Output			
P	Q	$(\mathrm{P} \rightarrow \mathrm{Q})$	$\rightarrow$	$(\mathrm{P} \rightarrow$	$(\mathrm{P} \wedge \mathrm{Q}))$
0	0	1	1	1	0
0	1	1	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Since $\mathrm{P} \rightarrow \mathrm{P}$ the truth of the major implication can be seen to be always `True`.

(b) Is the expression a tautology, a contradiction, or a contingency? Explain your answer.

Answer: The question is a tautology: It is always `True`.

On the other hand, a contradiction is never `True`, and a contingency can be `True` or `False` predicated on the value of the input.

Score

5. (10 pts) Consider the truth table below. Find a Boolean expression that computes it.

	Input			Output
Row	A	B	C	D
1	0	0	0	0
2	0	0	1	0
3	0	1	0	1
4	0	1	1	0
5	1	0	0	0
6	1	0	1	0
7	1	1	0	0
8	1	1	1	1

Answer: Take each row where the output D is 1. Form an AND-clauses of that input to make the output `True`. Form the OR of each of these clauses.

$$\mathrm{D} = (\neg \mathrm{A} \wedge \mathrm{B} \wedge \neg \mathrm{C}) \vee (\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{C})$$

Alternatively, take each row where the output D is 0. Form OR-clauses of that input to will make the output `False`. Form the AND of each of these clauses. We don't want to to that in this case, do we?

Score

6. (10 pts) Suppose you decide to write natural numbers using decimal digits. How many decimal numerals does it take to write the natural number n?

Answer: The approximate answer is $\log_{10} n$ where $\log_{10} \cdot$ is the logarithm base 10, often called the common logarithm.

The more precise answer is

$$\lfloor \log_{10} n \rfloor + 1 \quad \text{the floor of the log (base 10) of } n\text{, plus 1.}$$

Notice that any positive integer $m \geq 1$ can be bounded by powers of 10.

$$1 \leq m < 10, \quad 10 \leq m < 100, \quad 100 \leq m < 1000, \ldots, 10^{n-1} \leq m < 10^n$$

In these cases, the number m can be written in 1, 2, 3, $\ldots n$ digits.

Use these bounds and take the common logarithm of the inequalities to get

$$(n-1) \leq \log m < n$$

Round $\log m$ down to its floor, $(n-1)$, to compute the number of digits needed to write m.

$$n = \lfloor \log m \rfloor + 1$$

For different bases use different logarithm bases.

It is good to notice the similarities and differences between this problem and problem 1.

Score

7. (20 pts) Answer the following `True` or `False` questions. Explain your answers.

(a) $2 \cdot 2^n = 4^n$.

Answer: This is `False`. Please don't do this. It is irksome and shows reaction rather than thought: $2 \cdot 2^n = 2^{n+1}$.

A simple counterexample shows that it is `False`. Let $n = 0$. Then $2 \cdot 2^0 = 2 \neq 4^0 = 1$. The equation $2^{n+1} = 2 \cdot 2^n = 4^n = 2^{2n}$ is only `True` when $n + 1 = 2n$, that is $n = 1$.

(b) $\dfrac{n(n-1)}{2} + (n-1) = \dfrac{n(n+1)}{2}$.

Answer: This is `False`. The left-hand side of the equation is

$$\begin{aligned}\frac{n(n-1)}{2} + (n-1) &= (n-1)\left[\frac{n}{2}+1\right] \\ &= (n-1)\left[\frac{n+2}{2}\right] \\ &= \frac{(n-1)(n+2)}{2}\end{aligned}$$

(c) $\emptyset = \{\emptyset\}$

Answer: This is `False`. The empty set $\emptyset$ contains no elements. The set $\{\emptyset\}$ has one element, which happens to be $\emptyset$.

(d) $\emptyset \subseteq \emptyset$

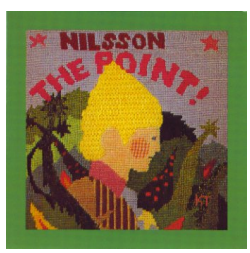
Answer: This is `True`. The empty set is a subset of every set, including itself.

Many students asked for an interpretation of $\subseteq$. I chose not to give one. Math is recognizing similarities, patterns. The symbol $\subseteq$ looks like $\leq$. How would you interpret <u>less than or equal</u> in the context of sets?

(e) $\mathbb{X} \cap \mathbb{Y} \subseteq \mathbb{X}$?

Answer: This is `True`. Every member of the intersection must also be in $\mathbb{X}$.

As in the previous problem, there are round things and pointy things. $\wedge$ and $\cap$ are similar except in their points. $\vee$ and $\cup$ are similar except in their points. Consider The Point! The story of Oblio the round headed boy banished from the Pointed Village to the Pointless Forest because he had no point.



Score

8. (10 pts) Prime, odd, and triangular numbers are interesting subsets of the natural numbers $\mathbb{N}$.

- <u>Prime</u> numbers have exactly two divisors. The prime digits are

$$\mathbb{P} = \{2, 3, 5, 7\}.$$

- <u>Odd</u> numbers have the form $2n+1$ for $n \in \mathbb{N}$. The odd digits are

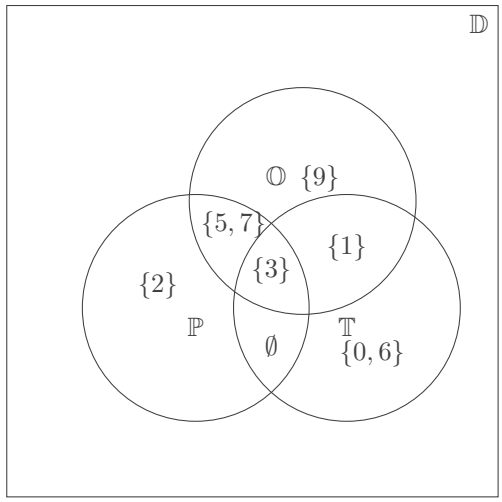
$$\mathbb{O} = \{1, 3, 5, 7, 9\}.$$

- <u>Triangular</u> numbers have the form $n(n-1)/2$ for $n \in \mathbb{N}$. The triangular digits are

$$\mathbb{T} = \{0, 1, 3, 6\}.$$

(a) Fill in the Venn diagram below using the elements $\mathbb{P}$, $\mathbb{O}$ and $\mathbb{T}$.

Answer:



(b) Shade the region $(\mathbb{P} \cap \mathbb{O}) \cup \mathbb{T}$.
Answer: