1. (10 pts) Fill in the truth table for De Morgan’s first law: If \( p \) and \( q \) are not both True, then one of \( p \) or \( q \) is False. Conversely, if one of \( p \) or \( q \) is False, then not both of them are True.

\[
\neg(p \land q) \equiv \neg p \lor \neg q
\]

De Morgan’s First Law

<table>
<thead>
<tr>
<th>Input</th>
<th>Computations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(q)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

2. (10 pts) Construct a truth table for the Boolean expression.

\[
[(p \to q) \land (\neg p \to r)] \to (q \lor r)
\]

Answer:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P)</td>
<td>(Q)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<td>0</td>
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<td>1</td>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
3. (10 pts) Given the truth table below, find a Boolean expression that computes it.

<table>
<thead>
<tr>
<th>Row</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0 0 1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0 1 0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0 1 1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1 0 0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1 0 1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1 1 0</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1 1 1</td>
<td>1</td>
</tr>
</tbody>
</table>

Answer: Taking the row where the output D is 1. Form an AND-clauses of the input to make the row True. Form the OR of these clauses.

\[ D = (\neg A \land B \land C) \lor (A \land \neg B \land C) \lor (A \land B \land \neg C) \lor (A \land B \land C) \]

Alternatively, take the row where the output D is 0. Form an OR-clauses of the input to make the row False. Form the AND of these clauses.

\[ D = (A \lor B \lor C) \land (A \lor B \lor \neg C) \land (A \lor \neg B \lor C) \land (\neg A \lor B \lor C) \]

4. (10 pts) The Internet Protocol (IP) is used to route traffic on the Internet. IPv6 uses 128-bits (16-bytes) to name addresses. What is the size of Internet version 6 space?

Answer: Internet version 6 space has \(2^{128} = 16^{32}\) addresses. A simple approximation gives

\[
2^{128} = (2^{10})^{12.8} \\
\approx (10^3)^{12.8} \\
= 10^{38.4}
\]

SI prefixes go up to yotta: \(10^{24}\). So there are about one hecto-tera-yotta-addresses \((10^{38})\) in IPv6 space.
5. (10 pts) Using \( n \) digits what is the largest natural number that can be written? Using \( n \) bits what is the largest natural number that can be written?

**Answer:** The largest decimal number that can written using \( n \) digits is \( 10^n - 1 \). The largest binary number that can written using \( n \) bits is \( 2^n - 1 \).

6. (10 pts) How many bits does it take to write the number 73? How many bits does it take to write the number \( n \)?

**Answer:** 73 can be written in 7 bits. Note that repeated division (quotients and remainders) is a simple way to compute a number’s binary representation. For instance,

<table>
<thead>
<tr>
<th>Repeated Remaindering Mod 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quotients</td>
</tr>
<tr>
<td>Remainders</td>
</tr>
</tbody>
</table>

Therefore 73 = \( (1001001)_2 \).

In general, any number \( n \geq 1 \) can be bound below and above by powers of 2

\[
2^k \leq n < 2^{k+1} \quad \text{for some } k \geq 0.
\]

In this case, it will require \( k + 1 \) bits to write \( n \). Using the inequalities above, and taking the log base 2 of each term, deduce that

\[
k \leq \lg n < k + 1
\]

Take the floor of \( \lg n \) to get \( k \) and add 1 to get \( k + 1 \), the number of bits needed to write \( n \).

\[
\lfloor \lg n \rfloor + 1 = \text{number of bits to write } n
\]
7. (30 pts) Let \( n, m \in \mathbb{N} \) be natural numbers. Write the following statements using mathematical notation.

(a) There is an \( n \in \mathbb{N} \) such that \( m = n/2 \).
Answer: \((\exists n \in \mathbb{N})(m = n/2)\)

(b) For all \( m \in \mathbb{N} \), there is an \( n \in \mathbb{N} \) such that \( n/2 = m \).
Answer: \((\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(m = n/2)\)

(c) There is an \( n \in \mathbb{N} \) such that for all \( m \in \mathbb{N} \), \( m = n/2 \).
Answer: \((\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m = n/2)\)

Which of these three statements are True, False, or undecidable.
Answer:
(a) Question (7a) is True: \( n = 2m \) is a natural number if \( m \) is and \( m = n/2 \).
(b) Question (7b) is True: Since there were no restrictions on \( m \) in the previous answer, the argument applies to all \( m \).
(c) Question (7c) is False: It says there is some fixed natural number \( n \) such that every other natural number is twice \( n \) \((m = 2n)\).

8. (10 pts) Okay, I have to know, who can solve my favorite quadratic equation?
\[x^2 - x - 1 = 0\]

The roots are interesting numbers, for 10 extra points, tell me about the roots.

Total Points: 100

Wednesday, January 22, 2014